Full extensions and approximate unitary equivalences *

Huaxin Lin
Department of Mathematics
University of Oregon
Eugene, Oregon 97403-1222

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Abstract

Let A be a unital separable amenable C^*-algebra and C be a unital C^*-algebra with certain infinite property. We show that two full monomorphisms $h_1, h_2 : A \to C$ are approximately unitarily equivalent if and only if $[h_1] = [h_2]$ in $KL(A, C)$. Let B be a non-unital but σ-unital C^*-algebra for which $M(B)/B$ has the certain infinite property. We prove that two full essential extensions are approximately unitarily equivalent if and only if they induce the same element in $KL(A, M(B)/B)$. The set of approximately unitarily equivalence classes of full essential extensions forms a group. If A satisfies the Universal Coefficient Theorem, it is can be identified with $KL(A, M(B)/B)$.

1 Introduction

The study of C^*-algebra extensions originated in the study of essentially normal operators on the infinite dimensional separable Hilbert space. The original Brown-Douglas-Fillmore theory gives a classification of essential normal operators via certain Fredholm related indices (see [7]). Later the Brown-Douglas-Fillmore theory gives classification of essential extensions of $C(X)$ by the compact operators (see [8], [5]). The study of C^*-algebra extensions developed into Kasparav's KK-theory and its application can be found not only in operator theory and operator algebras but also in differential geometry and noncommutative geometry.

Let $0 \to B \to E \to A \to 0$ be an essential extension of A by B. The extension is determined by a monomorphism $\tau : A \to M(B)/B$, the Busby invariant. When B is σ-unital stable C^*-algebra then $KK^1(A, B)$ gives a complete classification of these essential extensions –up to stable unitary equivalence. However, $KK^1(A, B)$ gives little information, if any, about unitary equivalence classes of the above mentioned extensions when $B \neq \mathcal{K}$ in general. There are known examples in which $KK^1(A, B) = \{0\}$ but inequivalent non-trivial extensions exist (see Example 0.6 of [24]). There are also known examples in which there are infinitely many inequivalent classes of trivial extensions (see 7.4 and 7.5 of [23]). When B is not stable,

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$KK^1(A, B)$ certainly should not be used to understand unitary equivalence classes of essential extensions mentioned above.

There are a number of results in classification of essential extensions (up to unitary equivalence or approximate unitary equivalence) when $B \neq \mathcal{K}$. Kirchberg's results ([18]) on extensions in which B is a non-unital purely infinite simple C^*-algebra shows that $KK^1(A, B)$ can be used to compute unitary equivalence classes of those extensions. When B is a non-unital but σ-unital simple C^*-algebra with continuous scale (see (6) below), then $M(B)/B$ is simple. Classification of essential extensions of a separable amenable C^*-algebra A by B (up to approximate unitary equivalence) was obtained in [32] (for some special cases in which $A = C(X)$, classification up to unitary equivalence was obtained in [22], [23] and [25]). In this case, B may not be stable, therefore $KK^1(A, B)$ is not used as invariant for essential extensions. Results about extensions of AF-algebras may be found in ([10],[16] and [14]).

In this paper, we study full essential extensions. These are essential extensions $\tau : A \to M(B)/B$ so that $\tau(a)$ is a full element for each nonzero element $a \in A$. Since the Calkin algebra $M(\mathcal{K})/\mathcal{K}$ is simple, all essential extensions by $\mathcal{K}$ are full. If B is a non-unital but σ-unital purely infinite simple C^*-algebra then $M(B)/B$ is also simple. Therefore essential extensions by those C^*-algebras are all full. The homogeneous extensions of A by $C(X) \otimes \mathcal{K}$ studied by Pimsner-Popa-Voiculescu ([35] and [36]) are all full extensions. In all these three cases, B is stable. There are non-stable, non-unital but σ-unital C^*-algebras which have continuous scale. In that case essential extensions by these C^*-algebras are also full. Furthermore, if A is a unital simple C^*-algebra and if the monomorphism $\tau : A \to M(B)/B$ is unital, then the essential extension induced by τ is always full for any non-unital C^*-algebra B.

With a technical condition on $M(B)/B$, we show that two full essential extensions are approximately unitarily equivalent if they induce the same element in $KL(A, M(B)/B)$ (see Theorem 2.8) provided that A is amenable and separable. When A is assumed to satisfy the so-called (Approximate) Universal Coefficient Theorem, we show that there is a bijective correspondence between approximate unitary equivalence classes of essential full extensions and $KL(A, M(B)/B)$. The advantage of study these full extensions is that full extensions (in these cases) are "approximately absorbing". For stable B, we show that $KK^1(A, B)$ classifies the unitary equivalence classes of essential full extensions. In this case, full extensions are "purely large" in the sense of Elliott and Kucerovsky ([15]).

The paper is organized as follows. Section 2 describes the main results in this paper. Section 3 shows that for many stable C^*-algebras their corona algebras $M(B)/B$ satisfy the technical condition (P1), (P2) and (P3). In Section 4, we show that there are examples of non-stable, non-unital and σ-unital C^*-algebras B for which $M(B)/B$ has the property (P1), (P2) and (P3). In Section 5, we give some modified results concerning amenable contractive completely positive linear maps. In Section 6, we discuss certain commutants in the ultrapower of corona algebras. In Section 7, we prove Theorem 2.8 mentioned above. In Section 8, we prove other main results described in Section 2.

We will use the following convention:

(1) Ideals in this paper are always closed and two-sided.

(2) Let A be a C^*-algebra and $p, q \in A$ be two projections. We write $p \sim q$ if there exists $v \in A$ such that $v^*v = p$ and $vv^* = q$.

(3) Let A and B be C^*-algebras and $L_1, L_2 : A \to B$ be linear maps. Let $\mathcal{F} \subset A$ and $\varepsilon > 0$, we write $L_1 \sim_\varepsilon L_2$ on $\mathcal{F}$, if

$$\|L_1(a) - L_2(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

(4) Let A and B be C^*-algebras. A contractive completely positive linear map $L : A \to B$ is said to *amenable*, if for $\varepsilon > 0$, there exists an integer $n > 0$ and two contractive completely positive linear maps $\phi : A \to M_n$ and $\psi : M_n \to A$ such that

$$\psi \circ \phi \sim_\varepsilon L \text{ on } \mathcal{F}.$$

(5) A C^*-algebra A is said to be *amenable* (or nuclear) if id_A is amenable.

(6) Let B be a non-unital but σ-unital simple C^*-algebra. B is said to have continuous scale, if there exists an approximate identity $\{e_n\}$ of B with $e_{n+1}e_n = e_n$ such that, for each nonzero element $b \in B$, there exists an integer $n > 0$ for which $e_{n+m} - e_n \lesssim b$ for all m (see [31]).

Let $e \in B$ be a nonzero projection and $T_e(B)$ be the set of all traces t on B for which $t(e) = 1$. Let B be a separable non-unital simple C^*-algebra with real rank zero, stable rank one and weakly unperforatated $K_0(B)$. If $\sup_n\{t(e_n)\}$ is a continuous function on $T_e(B)$, then B has continuous scale.

(7) Let $\{A_n\}$ be a sequence of C^*-algebras. Denote by $c_0(\{A_n\})$ the (C^*-) direct sum of $\{A_n\}$ and denote by $l^\infty(\{A_n\})$ the (C^*-) product of $\{A_n\}$. We use $q_\infty(\{A_n\})$ for the quotient $l^\infty(\{A_n\})/c_0(\{A_n\})$. When $A = A_n$ for all n, we write $c_0(A)$, $l^\infty(A)$ and $q_\infty(A)$ for simplicity.

(8) For each integer $n > 0$, define $f_n \in C_0((0, \infty))$ as follows

$$f_n(t) = \begin{cases} 1 & \text{if } t \geq 1/n; \\ \text{linear} & \text{if } 1/(n+1) \leq t < 1/n; \\ 0 & \text{if } 0 \leq t < 1/(n+1). \end{cases} \quad \text{(e 1)}$$

(9) An element a in a C^*-algebra A is said be *full*, if the ideal generated by a is A itself. Let A and B be two C^*-algebras and let $h : A \to B$ be a monomorphism. The monomorphism h is said to be *full* if $h(a)$ is full for every nonzero $a \in A$.

(10) Let $a \in A_+$ be a nonzero element, we write $Her(a)$ for the hereditary C^*-subalgebra $\overline{aAa}$ generated by a.

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2 Main results

Definition 2.1. Let B be a unital C^*-algebra. We say that B has property (P1) if for every full element $b \in B$ there exist $x, y \in B$ such that $xby = 1$. If b is positive, it is easy to see that $xby = 1$ implies that there is $z \in B$ such that $z^*bz = 1$.

It is obvious that an element b is full if and only if b^*b is full. It follows that B has property (P1) if and only if for every full element $0 \leq b \leq 1$, there exists $x \in B$ such that $x^*bx = 1$.

Every unital purely infinite simple C^*-algebra has the property (P1).

It turns out that many other unital C^*-algebras have the property (P1). Let A be a unital C^*-algebra and $B = A \otimes \mathcal{K}$. In next section we will show that $M(B)$ and $M(B)/B$ have property (P1) for many such that B. In Section 3, we will show that, for some non-stable (but σ-unital) C^*-algebra C, $M(C)$ and $M(C)/C$ may also have property (P1).

Definition 2.2. Let B be a unital C^*-algebra. We say that B has property (P2), if 1 is proper infinite, i.e., there is a projection $p \neq 1$ and partial isometries $w_1, w_2 \in B$ such that $w_1^* w_1 = 1$, $w_1 w_1^* = p$, $w_2^* w_2 = 1$ and $w_2 w_2^* \leq 1 - p$.

It is easy to see that, for each integer $n \geq 2$ and there are mutually orthogonal and mutually equivalent projections $s_{11}, s_{22}, ..., s_{nn}$ such that $1_B \geq \sum_{i=1}^n s_{ii}$ and there exists an isometry $Z \in B$ such that $Z^* Z = 1_B$ and $ZZ^* = s_{11}$. Let $C = s_{11} B s_{11}$. Then we may write $M_n(C) \subset B$.

It is clear that if B is stable then $M(B)$ and $M(B)/B$ have property (P2).

Proposition 2.3. *Let B be a unital C^*-algebra which has property (P1). Suppose that B contains two mutually orthogonal full elements. Then B has property (P2).*

Proof. Let $0 \leq a, b \leq 1$ be two mutually orthogonal full elements in B. Since B has property (P1), there are $x, y \in B$ such that $x^* a x = 1$ and $y^* b y = 1$. Let $v_1 = a^{1/2} x$ and $v_2 = b^{1/2} y$. Then $v_i^* v_i = 1$ and $s_{11} = v_1 v_1^*$ and $s_{22} = v_2 v_2^*$ are two projections. Thus B has property (P2). $\square$

Every purely infinite C^*-algebra (not necessary simple; see [19]) has property (P1) and (P2).

Definition 2.4. Let B be a unital C^*-algebra. We say that B has property (P3), if for any separable C^*-subalgebra $A \subset B$, there exists a sequence of sequences of elements $\{\{a_n^{(i)}\} : i = 1, 2, ...\}$ in B with $0 \leq a_n \leq 1$ such that

$$\lim_{n\to\infty} \|a_n^{(i)} c - c a_n^{(i)}\| = 0 \text{ for all } c \in A, \, i = 1, 2, ...,$$

$\lim_{n\to\infty} \|a_n^{(i)} a_n^{(j)}\| = 0$ if $i \neq j$ and for each i, and $\{a_n^{(i)}\}$ is a full element in $l^\infty(B)$.

Even though property (P3) looks more complicated than (P1) and (P2), it will be shown (see 3.13 below) that $M(B)/B$ has property (P3) for all $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra and for all B which have continuous scale and for many other non-unital σ-unital C^*-algebras B.

Proposition 2.5. *Let $B = C \otimes C_1$, where C_1 is a unital separable amenable purely infinite simple C^*-algebra. Then B has property (P1), (P2) and (P3).*

Let B be a non-unital but σ-unital C^*-algebra and A be a unital separable amenable C^*-algebra. We study essential extensions of the following form:

$$0 \to B \to E \to A \to 0. \tag{e 2}$$

Using the Busby invariant, we study monomorphisms $\tau : A \to M(B)/B$. We will only consider the case that the corona algebra $M(B)/B$ has the property (P1),(P2) and (P3).

Definition 2.6. An essential extension $\tau : A \to M(B)/B$ is said to be *full*, if τ is a full monomorphism . An extension τ is *weakly unital* if τ is unital monomorphism. If A is a unital simple C^*-algebra then every weakly unital essential extension is full. If $M(B)/B$ is simple, then every essential extension is full.

Definition 2.7. Let A be a unital separable C^*-algebra and C be a unital C^*-algebra. Suppose that $h_1, h_2 : A \to C$ are two homomorphisms. We say h_1 and h_2 are *approximately unitarily equivalent* if there exists a sequence of partial isometries $u_n \in C$ such that $u_n^* h_1(1_A) u_n = h_2(1_A)$, $u_n h_2(1_A) u_n^* = h_1(1_A)$ and

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ h_1(a) - h_2(a)\| = 0 \text{ for all } a \in A.$$

Note that if both h_1 and h_2 are unital, u_n can be chosedn to be unitaries.

Let B be a non-unital but σ-unital C^*-algebra. Two essential extensions of A by B are said to be approximately unitarily equivalent if the corresponding Busby invariants $\tau_1, \tau_2 : A \to M(B)/B$ are approximately unitarily equivalent.

Recall that $\tau : A \to M(B)/B$ is trivial if there is a monomorphism $h : A \to M(B)$ such that $\pi \circ h = \tau$, where $\pi : M(B) \to M(B)/B$ is a quotient map. In the case that $B = C \otimes \mathcal{K}$, where C is a σ-unital C^*-algebra τ_1 and τ_2 are *stably* unitarily equivalent if there exists a trivial extension $\tau_0 : A \to M(B)/B$ and a unitary $u \in M_2(M(B)/B)$ such that $\mathrm{ad} u \circ (\tau_1 \oplus \tau_0) = \tau_2 \oplus \tau_0$.

Let $\mathbf{Ext}(A, B)$ be the *stable* unitary equivalence classes of extensions of the form (e 2). When A is a separable amenable C^*-algebra $\mathbf{Ext}(A, B)$ may be identified with $KK^1(A, B)$. When A satisfies the Universal Coefficient Theorem, $KK^1(A, B)$ may be computable. However, as mentioned in the introduction, $KK^1(A, B)$ may not provide any useful information about unitary equivalence of extensions in general. In particular, when B is not stable, $KK^1(A, B)$ should not be used to describe unitary equivalence classes of essential extensions.

The first main result of this paper is the following:

Theorem 2.8. *Let A be a unital separable amenable C^*-algebra and B be a non-unital but σ-unital C^*-algebra so that $M(B)/B$ has the property (P1), (P2) and (P3). Suppose that $\tau_1, \tau_2 : A \to M(B)/B$ are two full monomorphisms. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \ \text{ in } \ KL(A, M(B)/B).$$

We will describe $KL(A, C)$ in 7.1. Theorem 2.8 is an easy corollary of the following theorem.

Theorem 2.9. *Let A be a unital separable amenable C^*-algebra and B be a unital C^*-algebra which has property (P1), (P2) and (P3). Suppose that $h_1, h_2 : A \to B$ are two full monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent, i.e, there exists a sequence of partial isometries $u_n \in B$ such that $u_n^* u_n = h_1(1_A)$, $u_n u_n^* = h_2(1_A)$ and*

$$\lim_{n \to \infty} \mathrm{ad}\, u_n \circ h_1(a) = h_2(a) \ \text{ for all } \ a \in A$$

if and only if $[h_1] = [h_2]$ in $KL(A, B)$.

Corollary 2.10. *Let A be a unital separable amenable simple C^*-algebra and B be a non-unital but σ-unital C^*-algebra so that $M(B)/B$ has the property (P1), (P2) and (P3). Suppose that $\tau_1, \tau_2 : A \to M(B)/B$ are two weakly unital essential extensions. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \ \text{ in } \ KL(A, M(B)/B).$$

Definition 2.11. *Let A be a unital separable amenable C^*-algebra and B be a unital C^*-algebra which has property (P2). Fix a full monomorphism $j_o : A \to \mathcal{O}_2 \to B$. Note (P2) implies such full monomorphisms do exist. Let $h_1, h_2 : A \to B \otimes \mathcal{K}$ be two homomorphisms. We write $h_1 \sim h_2$ if $h_1 \oplus j_o$ is approximately unitarily equivalent to $h_2 \oplus j_o$. Denote by $H(A, B)$ be "$\sim$ " equivalent classes of those homomorphisms.*

Proposition 2.12. *Let A be a unital separable amenable C^*-algebra and B be a unital C^*-algebra which has property (P2). Then $H(A, B)$ is a group with the zero element $[j_o]$.*

Corollary 2.13. *Let A be a unital separable amenable C^*-algebra and B be a unital C^*-algebra which has property (P1), (P2) and (P3). Let $H_f(A,B)$ be the approximate unitary equivalence classes of full monomorphisms from A to $B \otimes \mathcal{K}$. Then $H_f(A,B)$ is a group with the zero element $[j_o]$.*

Definition 2.14. Let A be a unital separable amenable C^*-algebra and B be a non-unital but σ-unital C^*-algebra. Denote by $\mathbf{Ext}^f_{ap}(A,B)$ the approximate unitary equivalence classes of full essential extensions. Denote by $\tau_o : A \to M(B)/B$ an essential extension which factors through $\mathcal{O}_2$. Note that $[\tau_o] = 0$ in $KL(A, M(B)/B)$. Suppose that $M(B)/B$ has property (P1), (P2) and (P3). It follows this τ_o is unique up to approximately unitary equivalence, by 2.10. Let $\tau_1, \tau_2 : A \to M(B)/B$ be two essential full extensions. Since $M(B)/B$ has property (P2), there are partial isometries $z_1, z_2 \in M(B)/B$ such that $z_1^* z_1 = 1_{M(B)/B}$, $z_1 z_1^* = \tau_1(1_A)$, $z_2^* z_2 = 1_{M(B)/B}$ and $z_2 z_2^* = \tau_2(1_A)$. Define $[\tau_1] + [\tau_2] = [\mathrm{ad}\, z_1 \circ \tau_1 \oplus \mathrm{ad}\, z_2 \circ \tau_2]$.

Note this is well defined, since $[\tau_o] = 0$ in $KL(A.M(B)/B)$ and $\mathrm{ad}\, z_1 \circ \tau \oplus \mathrm{ad}\, z_2 \circ \tau_o$ is approximately unitarily equivalent to τ by 2.10. With this addition $\mathbf{Ext}^f_{ap}(A,B)$ forms a semigroup. By 2.13, we have the following.

Corollary 2.15. *Let A be a unital separable amenable C^*-algebra and B be a non-unital but σ-unital C^*-algebra for which $M(B)/B$ has property (P1), (P2) and (P3). Then $\mathbf{Ext}^f_{ap}(A,B)$ is a group with zero element $[\tau_o]$, where $\tau_o : A \to M(B)/B$ is a full monomorphism which factors through $\mathcal{O}_2$.*

If furthermore, A satisfies so-called Approximate Universal Coefficient Theorem (AUCT) (see 7.1 below), then we have the following.

Theorem 2.16. *Let A be a unital separable amenable C^*-algebra which satisfies Approximate Universal Coefficient Theorem and B be a non-unital but σ-unital C^*-algebra so that $M(B)/B$ has the property (P1), (P2) and (P3). Then there is a bijection Γ from $\mathbf{Ext}^f_{ap}(A,B)$ onto $KL(A, M(B)/B)$.*

Approximate Universal Coefficient Theorem will be briefly discussed in 7.1 and 8.1. It should be noted that, when B is not stable, $K_i(M(B)/B)$ is very different from $K_i(SB)$, $i = 0, 1$. (see 1.7 of [32]).

In the special case that $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra, we have the following theorem:

Theorem 2.17. *Let A be a unital separable amenable C^*-algebra and $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra so that $M(B)/B$ has the property (P1). Suppose that $\tau_1, \tau_2 : A \to M(B)/B$ are two full essential extensions. Then τ_1 and τ_2 are unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \text{ in } KK^1(A,B).$$

Moreover, if $x \in KK^1(A,B)$, then there is a full essential extension $\tau : A \to M(B)/B$ such that $[\tau] = x$.

Theorem 2.18. *Let A be a unital separable amenable C^*-algebra and $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra for which the tracial state space $T(C) \neq \emptyset$. Suppose that there is $d > 0$ for which C satisfies the following:*

(1) *if $p, q \in B$ are two projections then $t(p) > d + t(q)$ for all $t \in T(C)$ implies $q \sim p$ in B;*

(2) *if $1 \geq b \geq 0$ in $M_k(C)$ such that $\tau(b) > \alpha + d$ for all $\tau \in T(A)$, then there is a projection $e \in \overline{bM_k(A)b}$ such that $\tau(e) > \alpha$ for all $\tau \in T(A)$.*

Then two essential full extensions $\tau_1, \tau_2 : A \to M(B)/B$ are unitarily equivalent if and only if

$$[\tau_1] = [\tau_2].$$

Remark 2.19. In the case that $B = \mathcal{K}$, Theorem 2.17 is the classical Brown-Douglas-Fillmore theorem. Note in this case, $M(\mathcal{K})/\mathcal{K}$ is a purely infinite simple C^*-algebra. It has property (P1) (as well as (P2) and (P3)) and every essential extension is full. Let X be a compact metric space with finite dimension d. When $B = C(X) \otimes \mathcal{K}$, $M(B)/B$ has property (P1) (see 3.9). Theorem 2.17 (or 2.18) deals with the extensions studied by Pimsner-Popa-Voiculescu (see [35] and [36]). When B is a non-unital purely infinite simple C^*-algebra this is obtained by Kirchberg. This theorem is closely related to a result of Elliott and Kucerovsky ([15]), see 8.7 for a discussion.

3 C^*-algebras have property (P1), (P2) and (P3)

Let A be a unital C^*-algebra. Denote by $T(A)$ (or T if no confusion exits) the set of tracial states on A. If $t \in T(A)$, we extend t to a trace $(t \otimes Tr)$ on $A \otimes M_n$ by defining $t((a_{ij}) = \sum_{i=1}^n t(a_{ii})$. We further use t for the trace defined on a dense set on $A \otimes \mathcal{K}$. If $a \in A \otimes \mathcal{K}_+$, then $t(a)$ is well defined (although it could be infinity). Suppose that $h_n \in A \otimes \mathcal{K}_+$ such that $h_n \nearrow h \in A^{**}$. Then one has $t(h) = \lim_{n\to\infty} t(h_n)$. These conventions will be used in this section.

The following lemma is certainly known

Lemma 3.1. *Let A be a unital C^*-algebra and I be a σ-unital ideal of A. If $a \in (A/I)_+$ is a full element, then there exists a full element $b \in A_+$ such that $\pi(b) = a$, where $\pi : A \to A/I$ is the quotient map.*

Proof. Since $a \in (A/I)_+$ is full, there are $x_1, x_2, ..., x_m \in A/I$ such that

$$\sum_{i=1}^m x_i^* a x_i = 1. \tag{e 3}$$

It follows that there are $c \in A_+$ and $y_1, y_2, ..., y_m \in A$ such that $\pi(c) = a$ and $1 - \sum_{i=1}^m y_i^* c y_i \in I$. Let e be a strictly positive element of I. Put $b = c + e$. Denote by J the ideal generated by b. Since $b \ge c$ and $b \ge e$, both c and e are in J. It follows that $I \subset J$. Since $\sum_{i=1}^m y_i^* c y_i \in J$, it follows that $1 \in J$. Thus $J = A$. Therefore b is full. ☐

Corollary 3.2. *Let A be a unital C^*-algebra and I be a σ-unital ideal of A. If A has property (P1), then so does A/I.*

Lemma 3.3. *Let A be a unital C^*-algebra $B = A \otimes \mathcal{K}$, Suppose that $a \in M(B)$ is an element for which $b = \pi(a)$ is full in $M(B)/B$, where $\pi : M(B) \to M(B)/B$ is the quotient map. Then a is full in $M(B)$. Furthermore, $M(B)/B$ has property (P1) so does $M(B)$.*

Proof. There are $x_1, x_2, ..., x_m\, y_1, ..., y_m \in M(B)/B$ such that $\sum_{i=1}^m x_i b y_i = 1$. Then there are $w_1, w_2, ..., w_m$, $z_1, z_2, ..., z_m \in M(B)$ such that

$$1 - \sum_{i=1}^m w_i a z_i \in B.$$

Let $\{e_{ij}\}$ be a system of matrix units. Put $E_n = \sum_{i=1}^n e_{ii}$. Then $\{E_n\}$ is an approximate identity consisting of projections. It follows that there exits $n > 0$ such that

$$\|\sum_{i=1}^m (1 - E_n) w_i a z_i (1 - E_n) - (1 - E_n)\| < 1/2 \tag{e 4}$$

Thus there exists $s \in (1-E_n)M(B)(1-E_n)$ such that

$$\sum_{i=1}^{m} s^*(1-E_n)w_i a z_i(1-E_n)s = 1-E_n. \tag{e 5}$$

But there exists $V \in M(B)$ such that $V^*(1-E_n)V = 1$. Therefore a is full.

For the last statement, we take $m = 1$ in the above argument. $\square$

Proposition 3.4. *Let B be a unital purely infinite simple C^*-algebra. Then $M(B\otimes\mathcal{K})$ and $M(B\otimes\mathcal{K})/A\otimes\mathcal{K}$ have the property (P1).*

Proof. It follows from [43] that $M(B\otimes\mathcal{K})/B\otimes\mathcal{K}$ is purely infinite and simple. Therefore $M(B\otimes\mathcal{K})/B\otimes\mathcal{K}$ has the property (P1). It follows from 3.3 that $M(B\otimes\mathcal{K})$ has property (P1). $\square$

Theorem 3.5. *Let $B = A\otimes\mathcal{K}$, where A is a unital separable C^*-algebra for which $T(A) \neq \emptyset$. Let $d > 0$. Suppose A satisfies the following:*

(1) *if $p, q \in B$ are two projections then $t(p) > d + t(q)$ for all $t \in T(A)$ implies $q \sim p$ in B;*

(2) *if $1 \geq b \geq 0$ in $M_k(A)$ such that $\tau(b) > \alpha + d$ for all $\tau \in T(A)$ (and some $\alpha > 0$), then there is a projection $e \in \overline{bM_k(A)b}$ such that $\tau(e) > \alpha$ for all $\tau \in T(A)$.*

Then $M(B)$ and $M(B)/B$ have property (P1).

Proof. Let $b \in M(B)$ be a full element. Without loss of generality, we may assume that $0 \leq b \leq 1$. Let $\{e_{ij}\}$ be the system of matrix unit for $\mathcal{K}$ and $E_n = \sum_{k=1}^n e_{ii}$. It follows that $E_n b E_n$ converges to b in the strict topology. Furthermore $b^{1/2}E_n b^{1/2}$ increasingly converges to b in the strict topology.

Since b is full, there are $x_1, x_2, ..., x_m \in M(B)$ such that

$$\sum_{k=1}^{m} x_i^* b x_i = 1.$$

Let $\tau \in T(A)$ be a tracial state. We extend τ to B_+ and then $M(B)_+$ in a usual way. Let T be the set of all (densely defined) traces on $M(B)_+$ whose restrictions to A are tracial states. With the usual weak *-topology, T is a compact convex set.

Because $b^{1/2}x_i^*x_i b^{1/2} \leq \|x_i\|^2 b$, one has

$$\tau(x_i^* b x_i) = \tau(b^{1/2}x_i^* x_i b^{1/2}) \leq \|x_i\|^2\tau(b)$$

for all $\tau \in T(A)$ Therefore

$$\sum_{i=1}^{m}\tau(x_i^* b x_i) \leq (\sum_{i=1}^{m}\|x_i\|^2)\tau(b)$$

for all $\tau \in T(A)$. Since $\tau(1) = \infty$, it follows that $\tau(b) = \infty$. Because $b^{1/2}E_n b^{1/2} \nearrow b$, and because T is compact, by the Dini's theorem, $\tau(b^{1/2}E_n b^{1/2}) \to \infty$ uniformly on T. Since $\tau(E_n b E_n) = \tau(b^{1/2}E_n b^{1/2})$ for all $\tau \in T$, $\tau(E_n b E_n) \nearrow \infty$ uniformly on T. There is $n(1) \geq 1$ such that

$$\tau(E_{n(1)} b E_{n(1)}) > 1 + 2d \ \text{ for all } \ \tau \in T.$$

Let A_1 be the hereditary C^*-subalgebra of B generated by $E_{n(1)} b E_{n(1)}$. It follows from assumption (2) that there is a projection $p_1 \in A_1$ such that $\tau(p_1) > 1 + d$ for all $\tau \in T$. It follows that there is $v_1 \in B$ such that

$v_1^* v_1 \le p_1$ and $v_1 v_1^* = E_1$. There are non-negative continuous function $f, g \in C_0((0, 2\|b\|]$ such that $gf = f$ and

$$\|f(E_{n(1)} b E_{n(1)}) p_1 f(E_{n(1)} b E_{n(1)}) - p_1\| < 1/4.$$

It follows (see A8 [13]) that there is a projection $q_1 \in \overline{f(E_{n(1)} b E_{n(1)}) B f(E_{n(1)} b E_{n(1)})}$ such that q_1 is unitarily equivalent to p_1. Since $gf = f$, we conclude that $gq_1 = q_1$. By functional calculus, we see that there are $f_1 \in A_1$ such that

$$f_1 E_{n(1)} b E_{n(1)} f_1 = g.$$

Thus we obtain $z_1 \in E_{n(1)} B E_{n(1)}$ such that

$$z_1^* b z_1 = z_1^* E_{n(1)} b E_{n(1)} z_1 = E_1.$$

Note that $\tau((1 - E_{n(1)}) b E_{n(1)}) = \tau(b E_{n(1)}(1 - E_{n(1)})) = 0$. It follows that

$$\tau((1 - E_{n(1)}) b (1 - E_{n(1)})) = \tau((1 - E_{n(1)}) b).$$

Since $\tau(E_{n(1)} b E_{n(1)}) < \infty$, for all $\tau \in T$, we conclude that

$$\tau((1 - E_{n(1)}) b (1 - E_{n(1)}) = \infty \quad \text{for all} \quad \tau \in T.$$

From the above argument, we obtain $n(2) > n(1)$ and $z_2 \in (E_{n(2)} - E_{n(1)}) B (E_{n(2)} - E_{n(1)})$ such that

$$z_2^* b z_2 = z_2^* (E_{n(2)} - E_{n(1)}) b (E_{n(2)} - E_{n(1)}) z_2 = E_2 - E_1.$$

Continuing this fashion, we obtain a sequence $\{n(k)\}$ with $n(k+1) > n(k)$ and $z_k \in (E_{n(k+1)} - E_{n(k)}) B (E_{n(k+1)} - E_{n(k)})$ such that

$$z_k^* b z_k^* = z_k^* (E_{n(k+1)} - E_{n(k)}) b (E_{n(k+1)} - E_{n(k)}) z_k = E_{k+1} - E_k,$$

$k = 1, 2, \ldots$. It follows that $z = \sum_{k=1}^{\infty} z_k \in M(B)$ since the sum converges in the strict topology. Furthermore we have

$$z^* b z = 1.$$

This shows that $M(B)$ has the property (P1).

By 3.2, $M(B)/B$ also has property (P1). □

From 3.5, we have the following corollaries.

Corollary 3.6. *Let A be a unital AF-algebra and $B = A \otimes \mathcal{K}$. Then $M(B)$ and $M(B)/B$ have property (P1).*

Proof. Clearly A satisfies (1) in 3.5 with any $d > 0$. To see that A satisfies (2), we let $1 \ge b \ge 0$ be an element in $M_n(A)$ such that $\tau(b) > \alpha + d$ for all $\tau \in T$. Let $C = \overline{b M_n(A) b}$ and let $\{e_n\}$ be an approximate identity for C consisting of projections. Then $\|e_n b e_n - b\| \to 0$ as $n \to \infty$. Since $0 \le b \le 1$, it follows that $\tau(e_n) > \alpha + d$ for some $n > 0$ and all $\tau \in T$. □

The proof of the corollary implies the following:

Corollary 3.7. *Let A be a unital separable C^*-algebra for which $T(A) \neq \emptyset$ and which satisfies (1) in 3.5 and has real rank zero. Then $M(B)$ and $M(B)/B$ have property (P1), where $B = A \otimes \mathcal{K}$.*

Corollary 3.8. *Let $B = A \otimes \mathcal{K}$, where A is a unital simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ Then both $M(B)$ and $M(B)/B$ have the property (P1).*

Corollary 3.9. *Let $A = C(X)$, where X is a compact Hausdorff space with covering dimension d. Then $M(A \otimes \mathcal{K})$ and $M(A \otimes \mathcal{K})/A \otimes \mathcal{K}$ have property (P1).*

Proof. Suppose that $e, f \in A \otimes \mathcal{K}$ are two projections. It is clear that we may assume that $e, f \in M_n(C(X))$ for some integer $n > 0$. Suppose that $\tau(e) > \tau(f) + d + 1$ for all $t \in T(A)$. It follows that for each $x \in X$, the rank of $e(x)$ is greater than $d + 1 +$ the rank of $f(x)$. It follows from 8.1.2 and 8.1.6 in [17] (see 6.10.3 (d) of [2]) that $f \lesssim e$. So (1) in 3.5 holds (for $(d+1)/2$).

For (2), let $1 \geq b \geq 0$ be an element in $M_k(C(X))$ for which $\tau(b) > \alpha + (d+1)$. Let f_n be as in (e 1). It follows that for some large n, $\tau(f_n(b)) > \alpha + (d+1)$ for all $\tau \in T(A)$. Thus, for each $\xi \in X$, the rank of $f_n(b)(\xi)$ is at least $\alpha + (d+1)$. By Lemma C in [4], there is a projection $e \in \overline{bM_k(A)b}$ such that the rank of $e(\xi)$ is greater α for all $\xi \in X$. It follows that $\tau(e) > \alpha$ for all $\tau \in T(A)$. □

To discuss property (P2), we begin with the following easy observation.

Proposition 3.10. *Let B be a unital C^*-algebra which has property (P2). Then, for any integer $n > 0$, there are $s_{11}, s_{22}, ..., s_{nn}$ such that $1_B \geq \sum_{i=1}^n s_{ii}$ and there exists an isometry $Z \in B$ such that $ZZ^* = e_{11}$. Moreover*

(1) *if for some $n \geq 2$, $1_B = \sum_{i=1}^n s_{ii}$, then there exists a unital embedding from $\mathcal{O}_n$ to B;*

(2) *there is a unital embedding from $\mathcal{O}_\infty$ to B and*

(3) *there exists a full embedding $j : \mathcal{O}_2 \to B$.*

Conversely, if there is a unital embedding from $\mathcal{O}_\infty$ to B, then B has property (P2). Furthermore, if B admits a full embedding from $\mathcal{O}_2$, then B has property (P2).

Proposition 3.11. (1) *Let A be a unital C^*-algebra and $B = A \otimes \mathcal{K}$. Then $M(B)$ and $M(B)/B$ has property (P2).*

(2) *Let A be a non-unital σ-unital simple C^*-algebra which has continuous scale. Then $M(A)/A$ has property (P2)*

(3) *Let A be a unital purely infinite simple C^*-algebra and $B = C_0(X, A)$, where X is a locally compact Hausdorff space. Then $M(B)$ and $M(B)/B$ have property (P2).*

Proof. For (3), we note there is a unital embedding from $\mathcal{O}_\infty$ to A and the constant maps from X into A are in $C^b(X, A) = M(B)$. □

Now we will turn to property (P3). Every unital purely infinite simple C^*-algebra has property (P3). This follows from 2.6 of [32]. Therefore, if B is a non-unital but σ-unital simple C^*-algebra with continuous scale, then $M(B)/B$ has property (P3).

Proposition 3.12. *Let B be a unital C^*-algebra which has the property (P1). Suppose that $0 \leq a, b \leq 1$ where $ab = a$ and a is full. Then there exists $x \in B$ with $\|x\| \leq 1$ such that*

$$x^* b x = 1. \qquad \text{(e 6)}$$

Note that the proposition includes the case that a is a full projection.

Proof. There is $z \in B$ such that $z^*az = 1$. Then $a^{1/2}zz^*a^{1/2} = p$ must be a projection. Moreover, $p \in Her(a)$. Therefore $pb = p$. Put $v = a^{1/2}z$. Then $v^*v = 1$ and $vv^* = p$. In particular, $\|v\| = 1$. Now

$$1 \geq \|b\|v^*v \geq v^*bv \geq v^*pv = 1. \tag{e 7}$$

We conclude that $v^*bv = 1$. □

Proposition 3.13. *Let A be a unital C^*-algebra, and $B = A \otimes \mathcal{K}$. Then $M(B)/B$ has property (P3).*

Proof. Let $\pi : M(B) \to M(B)/B$ be the quotient map and D be a separable C^*-algebra. Let $\{e_{i,j}\}$ be a system of matrix unit for $\mathcal{K}$. Denote by $E_n = \sum_{i=1}^n e_{i,i}$. It is known (see 3.12.14 of [33] and the proof of 5.5.3 of [27]) that there are $\{e_n\} \subset \mathrm{Conv}\{E_n : n = 1, 2, ..., \}$ such that

$$e_{n+1}e_n = e_n \quad \text{and} \quad \|e_n a - a e_n\| \to 0, \ \text{as} \ n \to \infty \tag{e 8}$$

for all $a \in D$.

Suppose that $e_n = \sum_{i=1}^{k(n)} \alpha_i E_i$, where α_i are non-negative scalars with $\sum_{i=1}^{k(n)} \alpha_i = 1$. There are $0 \leq \beta_j \leq 1$ such that $e_n = \sum_{i=1}^{k(n)} \beta_j e_{jj}$. Since, for each i,

$$\|e_m e_{ii} - e_{ii}\| \to 0 \quad \text{as} \ m \to \infty \tag{e 9}$$

there is $N(n) > 0$ such that, for each $m > N(n)$, $e_m = \sum_{i=1}^{k(m)} \beta_i e_{ii}$ with $\beta_{k(n)+1} > 1/2$. It follows that $(e_m - e_n)e_{k(n)+1,k(n)+1} = \beta_{k(n)+1} e_{k(n)+1,k(n)+1}$. By passing to a subsequence if necessary, without loss of generality, we may assume that $(e_{n+1} - e_n)e_{k(n)+1,k(n)+1} = \lambda_n e_{k(n)+1,k(n)+1}$ for some $\lambda_n > 1/2$. Now let $F \subset \mathbb{N}$ be an infinite subset. Then

$$b_F = \sum_{n \in F} (e_{n+1} - e_n) \geq (1/2) \sum_{n \in F} e_{k(n)+1,k(n)+1}. \tag{e 10}$$

It follows that b_F is a full positive element in $M(B)$. Suppose that $\{F_n\}$ is a sequence of infinite subsets of $\mathbb{N}$. Then, by 3.12, $\pi(\{\sum_{j \in F_n} e_{k(j)+1,k(j)+1})\}$ is full in $l^\infty(M(B)/B)$. So $\{\pi(b_{F_n})\}$ is full in $l^\infty(M(B)/B)$.

By (e 8), $\pi(b_F)$ commutes with $\pi(d)$ for each $d \in D$. Also by (e 8), if $|n - m| \geq 2$,

$$(e_{n+1} - e_n)(e_{m+1} - e_m) = 0. \tag{e 11}$$

It follows that $b_F b_{F'} = 0$, if $|n - m| \geq 2$ for any $n \in F$ and any $m \in F'$. Note that one may write $b_F = \sum_{n \in S(F)} \lambda_n e_{n,n}$, where each $0 < \lambda_n \leq 1$ is a positive number and $S(F)$ is an infinite subset of $\mathbb{N}$.

It is easy to find a family of (disjoint) infinite subsets $\{F_{i,j} : i, j = 1, 2, ...\}$ of $\mathbb{N}$ such that $|n - m| \geq 2$ for any $n \in S_{i,j}$ and any $m \in S_{i',j'}$, if $i \neq i'$, or $j \neq j'$, Define $S_{i,j} = S(F_{i,j})$ as above. We note that $S_{i,j} \cap S_{i',j'} = \emptyset$, if $i \neq i'$ or $j \neq j'$. Write $b_{i,j}$ for $b_{F_{i,j}}$. It follows that $M(B)/B$ has property (P3). □

4 Non-stable cases

In [19], Kirchberg and Rørdam extended the notion of purely infinite C^*-algebras to non-simple C^*-algebras. Let C_1 be a unital C^*-algebra and C_2 be a unital separable purely infinite simple C^*-algebra. Then $C_1 \otimes C_2$

is purely infinite (4.5 in [19]). Therefore, for any unital C^*-algebra C, $B = C \otimes \mathcal{O}_\infty$, has property (P1) and (P2) as well as (P3).

Proof of Proposition 2.5

Proof. By 4.5 in [19], B is purely infinite. It follows B has (P1) and (P2). Let A be separable C^*-subalgebra of B. There is a separable C^*-subalgebra $C_0 \subset C$ such that $A \subset C_0 \otimes C_1$. It follows from [20] that $C_1 \otimes \mathcal{O}_\infty \cong C_1$ and it follows from 7.2.6 of [40] and 3.12 of [20] that there is a sequence of unital monomorphisms $\phi_n : \mathcal{O}_\infty \to C_0 \otimes C_1$ such that

$$\lim_{n\to\infty} \|\phi_n(x)a - a\phi_n(x)\| = 0 \text{ for all } a \in C_0 \otimes \mathcal{C}_1. \tag{e 12}$$

Let $\{e_k\}$ be a sequence of nonzero mutually orthogonal projections in $\mathcal{O}_\infty$. Define $a_n^{(i)} = \phi_n(e_i)$, $n, i = 1, 2, \ldots$. One checks that $a_n^{(i)}$ satisfies the requirements in 2.4. □

There are σ-unital but *non-stable* separable C^*-algebras B for which the corona C^*-algebra $M(B)/B$ has property (P1), (P2) as well as (P3). For example, when B has continuous scale (see [21] and [31]) $M(B)/B$ is a purely infinite simple C^*-algebra (see [31]). So in those cases $M(B)/B$ has property (P1), (P2) as well as (P3). There are other non-stable separable C^*-algebras B for which B has property (P1), (P2) and (P3).

To make a point, we will present a very simple example of non-stable σ-unital C^*-algebra B for which $M(B)/B$ is not simple but both $M(B)$ and $M(B)/B$ have property (P1), (P2) and $M(B)/B$ has (P3).

It is clear that many such examples can be constructed.

Proposition 4.3 is not needed in Example 4.4 but will be used again later.

Lemma 4.1. *Let A be a unital C^*-algebra and $0 \le a \le 1$ be an element in A. Suppose that there is $x \in A$ such that $x^*ax = 1$. Then there is $N > 0$ depends on $\|x\|$ (not on A or a) for which there is $y \in A$ with $\|y\| \le 1$ such that*

$$y^* f_N(a) y = 1.$$

In particular, $f_N(a)$ is full (where f_N is as defined in (e 1)).

Proof. Let $q = a^{1/2}xx^*a^{1/2}$. Then q is a projection. There exists $k > 0$ depends on $\|x\|$ such that

$$\|f_k(t)t^{1/2} - t^{1/2}\| < \frac{1}{16\|x\|^2} \text{ for all } t \in [0, 1],$$

where f_k is as defined in (e 1). Then

$$\|f_k(a)q - q\| = \|(f_N(a)a^{1/2} - a^{1/2})x^*xa^{1/2}\| < 1/16.$$

It follows from A8 in [13] that there is a projection $p \in \overline{f_k(a)Af_k(a)}$ such that

$$\|q - p\| < 1/2.$$

Thus there exists $w \in A$ such that $w^*w = 1$ and $ww^* = p$. Choose $N = k + 1$. Then $f_N(a)q = q$. Thus

$$w^* f_N(a) w = 1.$$

□

Lemma 4.2. *Let A be unital C^*-algebra and $0 \le a \le 1$ be a full element in A. Suppose that there are $x_1, x_2, ..., x_m \in A$ such that*

$$\sum_{i=1}^{m} x_i^* a x_i = 1.$$

Let $r = \sum_{i=1}^{m} \|x_i\|^2$. Suppose also that $1_{M_m(A)} \lesssim 1$. Then there exists an integer $N > 0$ depends on r (but not A nor on a) such that $f_N(a)$ is full. Moreover, there are $y_1, y_2, ..., y_m \in A$ such that $\sum_{i=1}^{m} \|y_i\|^2 \le 1$ and

$$\sum_{i=1}^{m} y_i^* f_N(a) y_i = 1.$$

Proof. Let

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ 0 & 0 & \cdots & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots} \\ 0 & 0 & \cdots & a. \end{pmatrix}$$

Since $1_{M_m(A)} \lesssim 1$, one obtain $Y \in M_m(A)$ with $\|Y\| = 1$ such that $Y^* \mathrm{diag}(1, 0, ..., 0) Y = 1_{M_m(A)}$. Note we have $0 \le b \le 1$ and $XbX^* = \mathrm{diag}(1, 0, ..., 0)$. Thus

$$Y^* X b X^* Y = 1_{M_m(A)}.$$

We compute that $\|X^* Y\| \le r^{1/2}$. It follows from the above lemma that there is $N > 0$ $f_N(b)$ for which there is $z \in M_m(A)$ with $\|z\| \le 1$ such that

$$z^* f_N(b) z = 1_{M_m(A)}.$$

So $Y z^* f_N(b) z Y^* = 1$. An easy computation shows that there are $y_1, y_2, ..., y_n \in A$ such that $\sum_{i=1}^{m} \|y_i\|^2 \le 1$ and

$$\sum_{i=1}^{m} y_i^* f_N(a) y_i = 1.$$

□

Proposition 4.3. *Let $\{A_n\}$ be a sequence of unital C^*-algebras which has property (P1). Then $l^\infty(\{A_n\})$ also has property (P1).*

Proof. Let $a = \{a_n\}$ be a full element in $l^\infty(\{A_n\})$ such that $0 \le a \le 1$. By 4.2, there exists $N > 0$ such that $f_N(a)$ is full. For each n, there exists $x_n \in A_n$ such that $x_n^* f_N(a_n) x_n = 1$. Note that $f_{N+1}(a_n) f_N(a) = f_N(a)$. It follows from 3.12 that, for each n, there is $y_n \in A$ with $\|y_n\| \le 1$ such that

$$y_n^* f_{N+1}(a) y_n = 1.$$

Put $y = \{y_n\}$. Then $y \in l^\infty(\{A_n\})$. It is clear that there is $g \in C_0((0,1])_+$ such that

$$\|g(a) a g(a) - f_{N+1}(a)\| < 1/4.$$

Then

$$\|y^* g(a) a g(a) y - 1\| = \|y^*(g(a) a g(a) - f_N(a)) y\| \le 1/4.$$

It follows that there is $z \in l^\infty(\{A_n\})$ with $\|z\| < 4/3$ such that

$$z^* y^* g(a) a g(a) y z = 1.$$

□

The above proposition is not required in the following example. However it will be used in 6.5.

Example 4.4. Let A be a unital separable amenable purely infinite simple C^*-algebras. Denote by $B = c_0(A)$. Then $M(B) = l^\infty(A)$. Put $q_\infty(A) = l^\infty(A)/c_0(A)$. So $M(B)/B = q_\infty(A)$.

(1) $M(B)$ and $M(B)/B$ has property (P1) and (P2).

(2) $M(B)/B$ has property (P3).

It is clear that (1) is obvious (it also follows from 4.3). In fact, if $C = C_0((0,1), A)$, then $M(C)$ and $M(C)/C$ also have property (P1) and (P2). This could be proved rather easily.

To see (2), let D be a separable C^*-subalgebra of $M(B)$. Suppose that $x^{(1)} = \{x_n^{(1)}\}, x^{(2)} = \{x_n^{(2)}\}, ..., x^{(k)} = \{x_n^{(k)}\}, ...$ is a dense sequence of the unit ball of D. Using the fact that $A \otimes \mathcal{O}_\infty \cong A$ (see Theorem 3.15 of [20]), we obtain a sequence of homomorphisms $\phi_n : \mathcal{O}_\infty \to A$ such that

$$\lim_{n\to\infty} \|\phi_n(b)a - a\phi_n(b)\| = 0$$

for all $a \in A$ and $b \in \mathcal{O}_\infty$. Let $e_1 \in \mathcal{O}_\infty$ be a proper projection. There is an integer $n(1) > 0$ such that

$$\|\phi_{n(1)}(e_1)x_1^{(1)} - x_1^{(1)}\phi_{n(1)}(e_1)\| < 1/2.$$

There is a projection $e_2 \in \mathcal{O}_\infty$ such that $e_1e_2 = e_2e_1 = 0$ and $1 > e_1 + e_2$. There is $n(2) > 0$ such that

$$\|\phi_{n(2)}(e_j)x_l^{(i)} - x_l^{(i)}\phi_{n(2)}(e_j)\| < 1/4, \quad i, j, l = 1, 2.$$

Continuing in this fashion, we obtain a sequence of mutually orthogonal nonzero projections $\{e_m\} \subset \mathcal{O}_\infty$ and a subsequence $\{n(m)\}$ such that

$$\|\phi_{n(m)}(e_j)x_l^{(i)} - x_l^{(i)}\phi_{n(m)}(e_j)\| < 1/2^m, \quad i, j, l = 1, 2, ..., m.$$

Put $p^{(j)} = \{\phi_{n(m)}(e_j)\} \in l^\infty(A)$, $j = 1, 2, ...$ Then $p_m^{(i)}p_m^{(i)} = 0$ if $i \neq j$. Moreover,

$$\|\pi(p^{(j)})\pi(\{x^{(i)}\}) - \pi(\{x^{(i)}\})\pi(p^{(j)})\| = 0.$$

This implies that

$$\pi(p^{(j)})\pi(d) = \pi(d)\pi(p^{(j)}).$$

Put $a_n^{(j)} = p^{(j)}$, $j = 1, 2, ...$ This shows that $M(B)/B$ has property (P3).

It is clear, in fact, that $l^\infty(\{A_n\})/c_0(\{A_n\})$ has property (P3 if each A_n is a unital purely infinite simple C^*-algebra.

5 Amenable contractive completely positive linear maps

Lemma 5.1. (cf. 2.3 of [1], see also 5.3.2 of [27]) *Let A be a separable C^*-algebra and $\psi : A \to \mathbb{C}$ be a pure state. Denote also by ψ the extension of ψ on $\tilde{A}$ and put $L = \{a \in \tilde{A} : \psi(a^*a) = 0\}$. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is $z_i \in \tilde{A}_+$ with $\|z_i\| = 1$ such that $z_i \notin L$,*

$$z_{i+1}z_i = z_i, \ \ i = 1, 2, \quad \text{and} \quad \|z_i(\phi(a) - a)z_i\| < \varepsilon/2, \ \ i = 1, 2, 3, \tag{e 13}$$

for all $a \in \mathcal{F}$. Moreover, if $\{e_n\}$ is an approximate identity for A, then, for some large N,

$$\|e_nz_ie_n(\phi(a) - a)e_nz_ie_n\| < \varepsilon \quad \text{and} \quad e_nz_ie_n \notin L \tag{e 14}$$

for all $a \in \mathcal{F}$ and all $n \geq N$.

Proof. To simplify notation, we may assume that $\mathcal{F}$ is a subset of the unit ball of A. Let

$$N = \{a \in \tilde{A} : \phi(a) = 0\}.$$

Note that L is a closed left ideal. Let C be the hereditary C^*-subalgebra given by $L \cap L^*$. As in the proof of 5.3.2 in [27], we have $z_1, z_2, z_3, \in \tilde{A}$ with $\|z_i\| = 1$ $(i = 1, 2, 3)$ such that $z_i \notin L$, $z_{i+1}y_i = z_i$, $i = 1, 2, 3$, and

$$\|z_i(\psi(a) - a)z_i\| < \varepsilon/2, \;\; i = 1, 2, 3.$$

Let $\{e_n\}$ be an approximate identity for A such that $e_n e_{n+1} = e_n$ for all n. Note z_i has the form $\lambda_i 1_B + y_i'$, where $y_i' \in A$ and $\lambda_i \in \mathbb{C}$, $i = 1, 2$. Choose large n so that

$$\|e_k a - a e_k\| < \varepsilon/4, \;\; \|e_k a - a\| < \varepsilon/4 \quad \text{and} \quad \|e_k z_i - z_i e_k\| < \varepsilon/4$$

for all $a \in \mathcal{F} \cup \{z_1 a z_1, z_2 a z_2, z_3 z z_3 : a \in \mathcal{F}\}$ and for all $k \geq n$. Let $y_i = e_n z_i e_n$. Then, for $n \geq N$,

$$\|y_i(\psi(a) - a)y_i\| < \varepsilon/4 + \|e_n^2 z_i(\psi(a) - a)z_i e_n^2\| < \varepsilon \;\text{ for all }\; a \in \mathcal{F}.$$

□

The following is a folklore.

Lemma 5.2. *Let A be a C^*-subalgebra of B and $a \in A_+$. Denote by C the hereditary C^*-subalgebra of B generated by a. Then, for any approximate identity $\{e_n\}$ of A,*

$$\|e_n b - b\| \to 0 \quad \text{and} \quad \|b e_n - b\| \to 0, \;\text{ as }\; n \to \infty$$

for all $b \in C$.

Proof. There exists a sequence of positive function $f_n \in C_0(sp(a))$ with $0 \leq f_n \leq 1$ such that $\{f_n(a)\}$ forms an approximate identity for C. Fix an element $b \in C$. For any $\varepsilon > 0$, there is f_k such that

$$\|f_k(a)b - b\| < \varepsilon/4 \quad \text{and} \quad \|b f_k(a) - b\| < \varepsilon/4. \tag{e 15}$$

Choose integer $N > 0$ such that

$$\|e_n f_k(a) - f_k(a)\| < \frac{\varepsilon}{4(\|b\| + 1)} \;\text{ for all }\; n \geq N. \tag{e 16}$$

It follows that

$$\begin{aligned} \|e_n b - b\| &\leq \|e_n b - e_n f_k(a) b\| + \|e_n f_k(a) b - f_k(a) b\| + \|f_k(a) b - b\| && \text{(e 17)} \\ &< \varepsilon/4 + \|b\|\left(\frac{\varepsilon}{4(\|b\| + 1)}\right) + \varepsilon/4 = 3\varepsilon/4 < \varepsilon && \text{(e 18)} \end{aligned}$$

□

Lemma 5.3. *Let B be a unital C^*-algebra that has the property (P1). Let A be a separable C^*-algebra and I be an ideal of A. Suppose that $j : A \to B$ is an embedding such that $j(a)$ is a full element of B for all $a \notin I$. Then, for any pure state $\phi : A \to \mathbb{C}1_B \subset B$ which vanishes on I, any finite subset $\mathcal{F} \subset A$, and any $\varepsilon > 0$, there is a partial isometry $V \in B$ such that*

$$\|\phi(a) - V^* j(a) V\| < \varepsilon \;\text{ for }\; a \in \mathcal{F}, \;\; V^*V = 1_B \quad \text{and} \quad VV^* \in Her(j(A)).$$

Proof. To simplify notation, we identify A with $j(A)$. Fix $0 < \varepsilon < 1/2$. By 5.1, there are $z_1, z_2, z_3 \in \tilde{A}_+$ with $\|z_i\| = 1$ and $z_i \notin L$, $z_{i+1}z_i = z_i$, $i = 1, 2$ such that

$$\|\phi(a)z_i^2 - z_i j(a) z_i\| < \varepsilon/4 \ \text{ for } \ a \in \mathcal{F} \ \ (i = 1, 2) \tag{e 19}$$

for all $a \in \mathcal{F}$. Note $L = \{a \in \tilde{A} : \psi(a^*a) = 0\}$. Therefore $I \subset L \cap L^* \subset L$. Let $\{e_n\}$ be an approximate identity for A such that $e_n e_{n+1} = e_n$, $n = 1, 2....$ Let N be the integer as described in 3.12 so that

$$\|\phi(a)(e_n z_i e_n)^2 - e_n z_i e_n j(a) e_n z_i e_n\| < \varepsilon/2, \ \ i = 1, 2, 3. \tag{e 20}$$

Put $y_1 = e_N z_1 e_N$. We may assume that $y_1 \notin L$. By the assumption, y_1 is full. Because B has property (P1), there exists $x \in B$ such that $x^* y_1^2 x = 1_B$. Put $v_1 = y_1 x$. Then $v_1^* v_1 = 1_B$ and $v_1 v_1^* = p_1$ is a projection. Note that $p_1 \in Her(y_1)$. There is a projection in $q_1 \in Her(z_1^{1/2} e_N z_1^{1/2})$ such that q_1 is equivalent to p_1. Therefore there is a partial isometry $w_1 \in B$ such that $w_1^* q_1 w_1 = 1_B$ and $w_1 w_1^* = q_1$. Since $z_2^2 z_1 = z_1$, $z_2^2 q_1 = q_1$. By applying 5.2, one can choose a large integer $k > N$ so that, for all $n \geq k$,

$$\|e_n q_1 - q_1\| < \varepsilon/32 \quad \text{and} \quad \|e_n z_i - z_i e_n\| < \varepsilon/32, \ \ i = 1, 2, 3. \tag{e 21}$$

Thus

$$\|(e_k z_2 e_k)^2 q_1 - q_1\| = \|e_k z_2 e_k^2 z_2 e_k q_1 - q_1\| < 8\varepsilon/32 = \varepsilon/4. \tag{e 22}$$

Put $y_2 = e_k z_2 e_k$. Then one estimates

$$\|w_1^* y_2^2 w_1 - 1\| = \|w_1^* q_1 y_2^2 q_1 w_1 - w_1^* q_1 w_1\| < \varepsilon/2. \tag{e 23}$$

Thus there is $s \in Her(z_1^{1/2} e_N z_1^{1/2})_+ \subset B_+$ such that $\|s\| \leq \frac{1}{1-\varepsilon/2}$ and

$$s^{1/2} w_1^* y_2^2 w_1 s^{1/2} = 1. \tag{e 24}$$

Note that

$$\|w_1 s^{1/2}\| \leq \sqrt{\frac{1}{1-\varepsilon/2}} < \sqrt{\frac{2}{2-1/2}} = \sqrt{4/3} = \frac{2\sqrt{3}}{3}. \tag{e 25}$$

Define $V = y_2 w_1 s^{1/2}$. Note that

$$V^*V = 1_B \quad \text{and} \quad VV^* \in Her(j(A)). \tag{e 26}$$

Put $y_3 = e_{k+1} z_3 e_{k+1}$. Then, by (e 21),

$$\|y_3 y_2 - y_2\| = \|e_{k+1}(z_3 e_k z_2 - z_2) e_k\| < \varepsilon/32. \tag{e 27}$$

Furthermore, by (e 25) and (e 27),

$$\|y_3 V - V\| = \|y_3 y_2 w_1 s^{1/2} - y_2 w_1 s^{1/2}\| \leq (\varepsilon/32)(\frac{2\sqrt{3}}{3}) = \sqrt{3}\varepsilon/48. \tag{e 28}$$

We estimate, by applying (e 28),(e 26) and (e 20)

$$\begin{aligned} \|\phi(a) - V^*aV\| &= \|\phi(a)V^*V - V^*aV\| \\ &\leq 3\sqrt{3}\varepsilon/48 + \|\phi(a)V^*y_2^2V - V^*y_2 a y_2 V\| \\ &\leq 3\sqrt{3}\varepsilon/48 + \|\phi(a)y_2^2 - y_2 a y_2\| \\ &< 3\sqrt{3}\varepsilon/48 + \varepsilon/2 < \varepsilon \end{aligned} \tag{e 29}$$

for all $a \in \mathcal{F}$.

$\square$

Remark 5.4. If A has a unit, then the proof of Lemma 5.3 is almost identical to that of 5.3.2 of [27] which has its origin in [1]. When A has no unit, elements z_1, z_2, z_3 are not in A_+ but in $\tilde{A}_+$. By using an approximate identity $\{e_n\}$, one does have $\|y_3y_2 - y_2\|$ small. However the norm x could be large and depends on the choice of z_i as well as N as in the above proof. By introducing of q_1, we are able to control the norm of $w_1 s^{1/2}$.

Lemma 5.5. *Let B be a unital C^*-algebra which has the property (P1) and A be a separable C^*-algebra. Suppose that there exists a sequence of homomorphism $\phi_n : A \to B$ such that $\{\phi_n(a) : n = 1, 2, ...\}$ is a mutually orthogonal set in B for all $a \in A$. Let I be an ideal of A such that $\mathrm{ker}\phi_n \subset I$ and $\phi_n(a)$ is a full element in B for all $a \notin I$ for all n. Then, for any state $\psi : A/I \to \mathbb{C}1_B \subset B$, any finite subset $\mathcal{F} \subset A$, and any $\varepsilon > 0$, there is a partial isometry $V \in B$ and an integer n such that*

$$\|\psi \circ \pi(a) - V^*(\sum_{k=1}^{n} \phi_k(a))V\| < \varepsilon \ \text{ for } \ a \in \mathcal{F}, \ \ V^*V = 1B \quad \text{and} \quad VV^* \in Her(\sum_{i=1}^{n} \phi_i(A)), \tag{e 30}$$

where $\pi : A \to A/I$ is the quotient map. Moreover, if ψ is only assume to be a nonzero positive linear functional with $\|\psi\| \leq 1$, then the above still holds where V is merely a contraction.

Proof. By the Krein-Milman theorem, we have positive numbers $\alpha_1, \alpha_2, ..., \alpha_m$ with $\sum_{i=1}^m \alpha_i = 1$ and pure states $\psi_1, \psi_2, ..., \psi_m$ of A/I such that

$$\|\psi \circ \pi(a) - \sum_{i=1}^{m} \alpha_i \psi_i(a)\| < \varepsilon/2 \ \text{ for } \ a \in \mathcal{F}. \tag{e 31}$$

Let $\pi_n : A \to A/\mathrm{ker}\phi_n$ and $\gamma_n : A/\mathrm{ker}\phi_n \to A/I$ be the quotient maps, $n = 1, 2, ...$ Note that $\psi_i \circ \gamma_n$ is a pure state of $A/\mathrm{ker}\phi_n$.

By 5.5, there are $V_i \in B$ such that

$$V_i^* V_i = 1_B, V_i V_i^* \in Her(\phi_i(A)) \quad \text{and} \quad \|\psi_i \circ \gamma_i(\phi_i(a)) - V_i^* \phi_i(a) V_i\| < \varepsilon \tag{e 32}$$

for all $a \in \mathcal{F}$. One should note that

$$\psi_i \circ \gamma_i \circ \phi_i = \psi_i \circ \pi.$$

Set $V = \sum_{i=1}^m \sqrt{\alpha_i} V_i \in B$. We see that $V^*V = \sum_{i=1}^m \alpha_i 1_B = 1_B$ and $VV^* = \sum_{i=1}^m \alpha_i V_i V_i^* \in Her(\sum_{i=1}^m \phi_i(A))$. Moreover

$$\begin{aligned} \|\psi \circ \pi(a) - V^*(\sum_{i=1}^{m} \phi_i(a)V\| &= \|\psi \circ \pi(a) - \sum_{i=1}^{m} \alpha_i V_i^* \phi_i(a) V_i\| \\ &\leq \|\psi \circ \pi(a) - \sum_{i=1}^{m} \alpha_i \psi_i(a)\| + \sum_{i=1}^{m} \alpha_i \|\psi_i \circ \pi(a) - V_i^* \phi_i(a) V_i\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad a \in \mathcal{F}. \end{aligned} \tag{e 33}$$

To see the last statement of the lemma holds, we note that there is $0 < \lambda \leq 1$ such that $\psi(a) = \lambda \cdot g(a)$ for some state g and for all $a \in A$. $\square$

Lemma 5.6. *Let A be a separable C^*-algebra and B be a unital C^*-algebra which has the property (P1) and (P2). Let C be as described in 2.2 with $n = k$ (see 3.10). Suppose that $\phi_n : A \to B$ is a sequence of homomorphisms such that $\{\phi_n(a) : n = 1, 2, ...\}$ is a set of mutually orthogonal elements in B. Suppose that I is an ideal of A such that $I \supset \ker\phi_n$ and $\phi_n(a)$ is a full element for all $a \notin I$. Let $\psi : A/I \to M_k(\mathbb{C}) \subset M_k(C) \subset B$ be a contractive completely positive linear map Then for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a contraction $V \in B$ and an integer $m > 0$ such that*

$$\|\psi(a) - V^*(\sum_{i=1}^{m} \phi_i(a))V\| < \varepsilon \ \text{ for } \ a \in \mathcal{F} \quad \text{and} \quad VV^* \in Her(\sum_{i=1}^{K} \phi_i(A)), \tag{e 34}$$

where $\pi : A \to A/I$ is the quotient map.

Proof. Write $\psi(a) = \sum_{i=1}^{k} \psi_{ij}(a) \otimes s_{ij}$ for $a \in A$, where $\{s_{ij}\}$ is a system of matrix units for M_n and $\psi_{ij} : A \to \mathbb{C}$ is linear. Note we also assume that s_{ii} are as in 2.2 and 3.10, $i = 1, 2, ..., k$. Define $\Phi : M_k(A) \to \mathbb{C} \subset C$ by $\Phi((a_{ij})_{k\times k}) = \sum_{i,j=1} \psi_{ij}(a_{ij})$, where $a_{ij} \in A$. Let Z be as in 2.2 so that $ZZ^* = s_{11}$. Put $J_n(a) = Z_k\phi_n(a)Z^*$ for all $a \in A$. So J_n maps A into C $(= s_{11}Bs_{11})$. Note that $\phi_n \otimes \mathrm{id} : M_k(A) \to M_k(C)$ is also full. Set $\mathcal{G} = \{(a_{ij}) : a_{ij} \in \mathcal{F} \cup \{0\}\}$. Thus, by applying 5.5, there is $W \in M_k(B)$ with $\|W\| \le 1$ such that

$$\|\Phi(b) - W^*(\sum_{k=1}^{m} J_k \otimes \mathrm{id}(b))W\| < \varepsilon/2n^2 \ \text{ for } \ b \in \mathcal{G}. \tag{e 35}$$

Note that we may also assume that $W^*W \le \mathrm{diag}(1_C, 0, ..., 0)$. Choose a positive element $0 \le d \le 1$ in A such that

$$\|da - a\| < \varepsilon/2n^2 \ \text{ for all } \ a \in \mathcal{F}. \tag{e 36}$$

Let $v_i = (\overbrace{0, ..., 0}^{i-1}, d, 0, ..., 0)$ and v_i' be the $n \times n$ matrix with the first row as v_i and rest row are zero. Put $r_i = \sum_{n=1}^{m} J_n \otimes \mathrm{id}(v_i')$. Note that, for any $a \in A$,

$$\|r_i^*(\sum_{n=1}^{m} J_n(a) \otimes \mathrm{id}(a \otimes s_{11}))r_j - \sum_{n=1}^{m} J_n \otimes \mathrm{id}(a \otimes s_{ij})\| < \varepsilon/2n^2. \tag{e 37}$$

Therefore

$$\|\psi_{ij}(a) - W^*r_i^*(\sum_{k=1}^{m} J_k \otimes \mathrm{id}(a \otimes s_{11}))r_jW\| < \varepsilon/n^2 \tag{e 38}$$

for all $a \in \mathcal{F}$. Put $V' = (v_1'W, v_2'W, ..., v_n'W)$. Note we view V' is an $n \times n$ matrix with i-th column as a nonzero column of $v_i'W$, $i = 1, 2, ..., n$. Then

$$\|\psi(a) - V'^* \sum_{k=1}^{m} J_k \otimes \mathrm{id}(a \otimes e_{11})V'\| < \varepsilon \ \text{ for } \ a \in \mathcal{F}, \tag{e 39}$$

Define $V = Z^*V'$, we have

$$\|\psi(a) - V^* \sum_{n=1}^{m} \phi_n(a)V\| < \varepsilon \ \text{ for } \ a \in \mathcal{F}. \tag{e 40}$$

We also note that $VV^* \in Her(\sum_{n=1}^{m} \phi_n(A))$.

∎

Lemma 5.7. *Let A be a separable C^*-algebra and B be a unital C^*-algebra which has the property (P1) and property (P2). Suppose that $\phi_n : A \to B$ is a sequence of homomorphisms such that the embedding $j_n : \phi_n(A) \to B$ is full where $\{\phi_n(a) : n = 1, 2, ...\}$ is a set of mutually orthogonal elements in B. Suppose that $\psi : A \to B$ is amenable such that $\ker\psi \supset \ker\phi_n$, $n = 1, 2, ...$ Then, for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a contraction $V \in B$ and an integer $K > 0$ such that*

$$\|\psi(a) - V^*(\sum_{i=1}^{K} \phi_i(a))V\| < \varepsilon \text{ for } a \in \mathcal{F} \quad \text{and} \quad VV^* \in Her(\sum_{i=1}^{K} \phi_i(A)). \tag{e 41}$$

Proof. Fix a finite subset $\mathcal{F}$ and $\varepsilon > 0$. Since ψ is amenable, to simplify notation, without loss of generality, we may assume that $\psi = \alpha \circ \beta$, where $\beta : A \to M_n = M_n(\mathbb{C} \cdot 1_C)$ and $\alpha : M_n \to B$ are contractive completely positive linear maps (it should be noted though that n depends on $\mathcal{F}$ as well as ε). Write $M_n(C) \subset B$ as in 2.2 (see also 3.10). Put $\mathcal{G} = \beta(\mathcal{F})$. It is convenient to assume that $\mathcal{F}$ lies in the unit ball of A so $\mathcal{G}$ lies the unit ball of $M_n(\mathbb{C} \cdot 1_C)$. Note that $\sigma : M_n \to M_n(\mathbb{C}) \subset B$ is full. There exists an integer $m > 0$ and a contraction $Z \in M_m(B)$ such that

$$\|\alpha(b) - Z^* \mathrm{diag}\,\overbrace{(b, b, ..., b)}^{m} Z\| < \varepsilon/4 \text{ for } b \in \mathcal{G}. \tag{e 42}$$

It follows from 5.6 that there is $N(1) > 1$ and a contraction $W_1 \in B$ such that

$$\|\beta(a) - W_1^* \sum_{i=1}^{N(1)} \phi_i(a) W_1\| < \varepsilon/4m \text{ for } a \in \mathcal{F} \tag{e 43}$$

as well as integers $N(k+1) > N(k)$ and a contractions $W_k \in B$ such that

$$\|\beta(a) - W_{k+1}^* \sum_{i=N(k)+1}^{N(k+1)} \phi_i(a) W_{k+1}\| < \varepsilon/4m \text{ for } a \in \mathcal{F}, k = 1, 2, ...,. \tag{e 44}$$

Note we have

$$\|\alpha \circ \beta(a) - Z^* \mathrm{diag}\,\overbrace{(\beta(a), \beta(a), ..., \beta(a))}^{m} Z\| < \varepsilon/2 \text{ for } a \in \mathcal{F}. \tag{e 45}$$

It follows that

$$\|\psi(a) - Z^*(\mathrm{diag}(W_1^* \sum_{i=1}^{m(1)} \phi_i(a) W_1, \cdots, W_m \sum_{i=N(m-1)+1}^{N(m)} \phi_i(a)) W_m)) Z\| < \varepsilon/2$$

for all $a \in \mathcal{F}$. There exists $d_i \in Her(\phi_i(A))_+$ with $0 \le d_i \le 1$ such that

$$\|d_i \phi_i(a) - \phi_i(a)\| < \varepsilon/2m \quad \text{and} \quad \|d_i \phi_i(a) d_i - \phi_i(a)\| < \varepsilon/2m \tag{e 46}$$

for all $a \in \mathcal{F}$. Note that $d_i d_j = 0$ if $i \neq j$, $i, j = 1, 2, ..., m$. Now let Y be the $n \times n$ matrix so that the first row is $(d_1, d_2, ..., d_m)$ and the rest are zero. Put $W = \mathrm{diag}(W_1, W_2, ..., W_m)$ and $V = YWZ$. Then

$$\|\mathrm{diag}(W_1^* \sum_{i=1}^{m(1)} \phi_i(a) W_1, \cdots, W_m \sum_{i=N(m-1)+1}^{N(m)} \phi_i(a) W_m) - W^* Y^* \sum_{k=1}^{N(m)} \phi_k(a) YW\| < \varepsilon/2$$

for $a\in\mathcal{F}$. Moreover

$$\|\psi(a)-V^*\sum_{k=1}^{N(m)}\phi_k(a)V\|<\varepsilon \ \text{ for } \ a\in\mathcal{F} \quad\text{and}\quad VV^*\in Her(\sum_{k=1}^{N(m)}\phi_k(A)). \tag{e 47}$$

□

6 Commutants in the ultrapower of corona algebras

Definition 6.1. Recall that a family ω of subsets of $\mathbb{N}$ is an *ultrafilter* if

(i) $X_1,...,X_n\in\omega$ implies $\cap_{i=1}^n X_i\in\omega$,

(ii) $\emptyset\notin\omega$,

(iii) if $X\in\omega$ and $X\subset Y$, then $Y\in\omega$ and

(iv) if $X\subset\mathbb{N}$ then either X or $\mathbb{N}\setminus X$ is in ω.

An ultrafilter is said to be *free*, if $\cap_{X\in\omega}X=\emptyset$. The set of free ultrafilters is identified with elements in $\beta\mathbb{N}\setminus\mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Cech compactification of $\mathbb{N}$.

A sequence $\{x_n\}$ (in a normed space) is said to converge to x_0 along ω, written $\lim_\omega x_n=x_0$, if for any $\varepsilon>0$ there exists $X\in\omega$ such that $\|x_n-x_0\|<\varepsilon$ for all $n\in X$.

Let $\{B_n\}$ be a sequence of C^*-algebras. Fix an ultrafilter ω. The ideal of $l^\infty(\{B_n\})$ which consists of those sequences $\{a_n\}$ in $l^\infty(\{B_n\})$ such that $\lim_\omega\|a_n\|=0$ is denoted by $c_\omega(\{B_n\})$. Define

$$q_\omega(\{A_n\})=l^\infty(\{B_n\})/c_\omega(\{B_n\}).$$

If $B_n=B, n=1,2,...$, we use $c_\omega(B)$ for $c_\omega(\{B_n\})$ and $q_\omega(A)$ for $q_\omega(\{A_n\})$, respectively.

Lemma 6.2. *Let A be a C^*-algebra, I be an ideal of A and let $a\in A\setminus\{0\}$ such that $0\le a\le 1$. Suppose that $a\notin I$. Then there is $b\in C^*(a)$ with $0\le b\le 1$ and $\|b\|=1$ such that if $c\in C^*(b)\setminus J$, then $c\notin I$, where*

$$J=\{f(b): f\in C_0(sp(b)\setminus\{0\}),\ \ f(1)=0\}.$$

Proof. Let $\pi: A\to A/I$ be the quotient map. Then $\pi(a)\neq 0$. Suppose that $\xi\in sp(a)\setminus\{0\}$. Let $f\in C_0(sp(a)\setminus\{0\})$ such that $f(\xi)=1$ and $0<f(t)<1$ for all other $t\in sp(a)$. Set $b=f(a)$. Then, $\pi(b)\neq 0$. and $\|\pi(b)\|=1$. If $c\notin J$, $c=g(a)$ for some $g\in C_0(sp(a)\setminus\{0\})$ such that $g(\xi)\neq 0$. It follows that $\pi(g(a))\neq 0$. Therefore $c\notin I$. □

Lemma 6.3. *Let B be a unital C^*-algebra and $a\in B$ be an element with $0\le a\le 1$. Suppose that there is $x\in B$ such that $x^*ax=1$. Then there exists an element $b\in C^*(a)$ such that c is full for all $c\in C^*(b)\setminus J$, where*

$$J=\{f(b): f\in C_0(sp(b)\setminus\{0\}),\ \ f(1)=0\}.$$

Proof. Put $v=a^{1/2}x$. Then $v^*v=1$ and $vv^*=q$ for some projection $q\in B$. Note that $q\in Her(a^{1/2}xx^*a^{1/2})\subset Her(a)$. For any $0<\varepsilon/<1/4$, there is $N>0$ such that

$$\|f_n(a)p-p\|<\varepsilon/2 \ \text{ for all } \ n\ge N.$$

(f_n be as in (e 1).) It follows that

$$\|f_n(a)pf_n(a)-p\|<\varepsilon$$

for all $n \geq N$. If follows that there is a projection $q \in Her(f_N(a))$ and partial isometry $w \in B$ such that $w^*qw = 1$ and $ww^* = q$. Thus $f_{N+1}(a)q = q$. Put $b = f_{N+1}(a)$. Thus, for any function $g \in C_0((0,1])$, if $g(1) \neq 0$, then $g(b)q = q$. It follows that $w^*g(b)w = 1$. Thus $g(b)$ is full. The lemma follows. ☐

Lemma 6.4. *Let A be a unital separable C^*-subalgebra of a unital C^*-algebra B which has property (P1) and (P3). Suppose that every nonzero element in A is full in B. Then there exists a sequence of sequences of positive elements $\{a_n^{(i)}\}$, $i = 1, 2, ...$ with $0 \leq a \leq 1$ satisfying the following:*

(1) $\lim_{n\to\infty} \|a_n^{(i)}a - aa_n^{(i)}\| = 0$ *for all $a \in A$ and $i = 1, 2, ...$;*

(2) $\lim_{n\to\infty} \|a_n^{(i)}a_n^{(j)}\| = 0$ *if $i \neq j$ and*

(3) $\Pi(\{a_n^{(i)}\})\Pi \circ J(a)$ *is full in $q_\omega(A)$ for any free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, where $J : B \to l^\infty(B)$ is defined by $J(b) = (b, b, ..., b, ...)$ for $b \in B$ and $\Pi : l^\infty(B) \to q_\omega(B)$ is the quotient map.*

Proof. For each nonzero element $0 \leq a \leq 1$ in A, define

$$r(a) = \inf\{\|x\| : x^*ax = 1\}.$$

Let $b_1, b_2, ..., b_n, ...$ be a dense sequence of the unit ball of A. We may assume that $\{b_n\}$ contains a subsequence of positive elements which is dense in the positive part of the unit ball. For each $0 \leq b_k \leq 1$ in the sequence, from the assumption, there is $x_k \in B$ such that $x_k^*b_kx_k = 1$ and $\|x_k\| \leq (4/3)r(b_k)$. Let D be the separable C^*-subalgebra generated by A and $\{x_k\}$.

We claim that, for each nonzero $a \in A$ with $0 \leq a \leq 1$ there is $x \in D$ such that $x^*ax = 1$. There is $z \in B$ such that $z^*az = 1$ and $\|z\| < (4/3)r(a)$. There is b_k with $0 \leq b_k \leq 1$ for which

$$\|a - b_k\| < 1/8((4/3)r(a) + 1)^2.$$

Then

$$\|z^*b_kz - 1\| \leq \|z^*(b_k - a)z\| \leq 1/8.$$

We obtain $y \in D$ with $\|y\| < 8/7$ such that

$$y^*z^*b_kzy = 1.$$

It follows that $r(b_k) \leq (8/7)r(a)$. Hence there is $x_k \in D$ with $\|x_k\| \leq (4/3)(8/7)r(a)$ such that $x_k^*b_kx_k = 1$. It follows that

$$\|x_k^*ax_k - 1\| \leq \|x_k^*(a - b_k)x_k\| < (1/8((4/3)r(a) + 1)^2)[(4/3)(8/7)r(a)]^2 < 8/49 < 1.$$

Thus there is $d \in D$ such that

$$d^*x_kax_kd = 1.$$

This proves the claim.

Now since B has property (P3) and D is separable, there exists a sequence of sequences of nonzero elements $\{a_n^{(i)}\}$ in B with $0 \leq a_n^{(i)} \leq 1$ such that

(i) $\lim_{n\to\infty} \|a_n^{(i)}d - da_n^{(i)}\| = 0$ for all $d \in D$;

(ii) $\lim_{n\to\infty} \|a_n^{(i)}a_n^{(j)}\| = 0$ if $i \neq j$ and

(iii) for each i, $\{a_n^{(i)}\}$ is full in $l^\infty(A)$.

Thus (1) and (2) follow. To see (3), let $a \in A$. From the claim, there is $d \in D$ such that

$$d^*ad = 1.$$

Put $a_i = \{a_n^{(i)}\}$. Then, by 4.3, there is $z \in l^\infty(A)$ such that $z^*a_iz = 1$. Note (i) implies that

$$\Pi(a_i)\Pi \circ J(b) = \Pi \circ J(b)\Pi(a_i) \text{ for all } b \in D.$$

Put $g = \Pi \circ J(d)\Pi(z)$. Then

$$g^*\Pi(a_i)\Pi \circ J(a)g = \Pi(z^*)\Pi \circ J(d^*)\Pi(a_i)\Pi \circ J(a)\Pi \circ J(d)\Pi(z)$$

$$= \Pi(z^*)\Pi(a_i)\Pi \circ J(d^*)\pi \circ J(a)\Pi \circ J(d)\Pi(z) = \pi(z^*)\Pi(a_i)\Pi(z) = 1.$$

□

Lemma 6.5. *Let A be a unital separable amenable C^*-algebra, B be a unital C^*-algebra which has property (P1), (P2) and (P3). Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Suppose that $\tau : A \to B$ is a full unital embedding. Let $\tau_\infty : A \to l^\infty(B)$ be defined by $\tau_\infty(a) = (\tau(a), \tau(a), ...)$ and let $\psi = \Pi \circ \tau_\infty$, where $\Pi : l^\infty(B) \to q_\omega(B)$. Then there is a unital C^*-subalgebra $C \cong \mathcal{O}_\infty$ in the commutant of $\psi(A)$ in $q_\omega(B)$.*

Proof. Let $\{a_n^{(i)}\}$ be the sequence of sequences of elements given by 6.4. Put $a_i = \{a_n^{(i)}\}$, $i = 1, 2, ...$ Let D be as in the proof 6.4.

Applying 6.3 (and also using D as in the proof of 6.4), we may assume that each a_i has the property that $sp(a_i) \subset [0,1]$ and $f(\Pi(a_i))$ is full for all $0 \le f \le 1$ in $C_0((0,1])$ for which $f(1) \neq 0$.

Let $X = (0,1]$ and fix i. Define $\phi_j', L' : C_0(X) \otimes A \to q_\omega(B)$ by $\phi_i'(f \otimes a) = f(\Pi(a_i))\psi(a)$ and $L'(f \otimes a) = f(1)\psi(a)$ for $a \in A$, respectively. By (3) in 6.4, ϕ_i' is full. Let $\{\mathcal{F}_j\}$ be an increasing sequence of finite subsets of A for which $\cup_{n=1}^\infty \mathcal{F}_j$ is dense in A and $\{g_n\}$ be a dense sequence of $C_0((0,1])$.

Let $\{a_{i(k)}\}_{k=1}^\infty$ be a subsequence of $\{a_i\}$. It follows from 5.7 that there exists $s_n \in B$ such that

$$\|s_n^*(\sum_{k=1}^{m(n)} g_j(a_{i(k)})\psi(a))s_n - g_j(1)\psi(a)\| < 1/2^n \text{ for } a \in \mathcal{F}_n \quad \text{and} \quad j = 1, 2, ..., n \tag{e 48}$$

Moreover, $s_ns_n^* \in \mathrm{Her}(\sum_{k=1}^{m(n)}(a_{i(k)})\psi(A))$. Suppose that $s_n = \Pi((s_{n,1}, s_{n,2}, ...))$, $n = 1, 2, ...$ We may assume that

$$\|s_{n,k(n)}^*(\sum_{k=1}^{m(n)} g_j(a_{k(n)}^{(i(k)})\tau(a))s_{n,k(n)} - g_j(1)\tau(a)\| < 1/2^n, n = 1, 2, ...$$

Now put $t_n = s_{n,k(n)}$, $t' = (t_1, t_2, ...)$ and $t = \Pi(t')$. Define $\Phi : C_0(X) \otimes A \to l^\infty(B)$ by

$$\Phi(f \otimes a) = \{\sum_{k=1}^{m(n)} f(a_{k(n)}^{(i(k)})\tau(a)\} \text{ for all } f \in C_0(X) \quad \text{and} \quad a \in A.$$

It follows that

$$t^*\Pi \circ \Phi(f \otimes a)t = f(1)\psi(a) \text{ for all } f \in C_0(X) \quad \text{and} \quad a \in A.$$

Put $b(\{i(k)\}) = \Pi(\{a_{k(n)}^{i(k)}\})$. Note that $0 \le b(\{i(k)\}) \le 1$. We have (with $\imath(t) = t$ for all $t \in (0,1]$)

$$t^*b(\{i(k)\})t = \imath(1) = 1_{q_\omega(B)}.$$

Put $w(\{i(k)\}) = b(\{i(k)\})^{1/2}t$ and $q = b(\{i(k)\})^{1/2}tt^*b(\{(i(k)\})^{1/2}$. Since $b(\{i(k)\}) \in \psi(A)'$ and $\imath(1) = 1$, we have

$$t^*b(\{i(k)\})^{1/2}\psi(a)b(\{i(k)\})^{1/2}t = t^*b(\{i(k)\})\psi(a)t = \imath(1)\psi(a) = \psi(a) \ \text{for all}\ a \in A. \tag{e 49}$$

It follows from 6.36 in [40] that $w(\{i(k)\}) = b(\{i(k)\})^{1/2}t \in \psi(A)'$. It clear that if $\{i(k)\}$ and $\{i(k)'\}$ are two disjoint infinite subsets of $\mathbb{N}$, then corresponding projections q and q' are orthogonal. This implies that one has a sequence of isometries $v_k \in \psi(A)'$ such that $v_k^*v_k = 1_{q_\omega(B)}$ and $1 \geq \sum_{k=1}^n v_kv_k^*$, $n = 1, 2, ...$ Thus $\psi(A)'$ admits a unital embedding of $\mathcal{O}_\infty$, □

7 Full extensions

Definition 7.1. Let $\mathbf{Ext}(A, B)$ be the usual set of *stable* unitary equivalence classes of extensions of the form (e 2). When A is amenable, it is known (Arveson/Choi-Effros) that $\mathbf{Ext}(A, B)$ is a group. Moreover, it can be identified with $KK^1(A, B)$. Let $\mathcal{T}(A, B)$ be the set of all *stable* unitary equivalence classes of approximately trivial extensions. It is known that $\mathcal{T}(A, B)$ is a subgroup of $KK^1(A, B)$ (see [28]). Following Rørdam, one defines $KL^1(A, B) = KK^1(A, B)/\mathcal{T}(A, B)$.

Let G_i, $i = 1, 2, 3$ be three abelian groups. A group extension $0 \to G_1 \to G_3 \to G_2 \to 0$ is said to be *pure* if every finitely generated subgroup of G_2 lifts. Denote by $Pext(G_2, G_1)$ the set of all pure extensions and $E(G_2, G_1) = ext_{\mathbb{Z}}(G_2, G_1)/Pext(G_2, G_1)$.

If A satisfies the Approximate Universal Coefficient Theorem (AUCT) –see [28], then one has the following short exact sequence:

$$0 \to E(K_i(A), K_i(B)) \to KL^1(A, B) \to Hom(K_i(A), K_{i-1}(B)) \to 0. \tag{e 50}$$

So $KL^1(A, B)$ is computable in theory. It should be noted every separable amenable C^*-algebra which satisfies the Universal Coefficient Theorem (UCT) satisfies the AUCT. Rosenberg and Schochet ([37]) show that every separable C^*-algebras in the so-called "bootstrap" class satisfies the UCT (therefore the AUCT). We also use the notation $KL(A, B) = KL^1(A, SB)$.

As mentioned in the introduction, two stably unitarily equivalent extensions are in general not unitarily equivalent and trivial extensions are not unitarily equivalent. Furthermore, an essential extension which is zero in $KK^1(A, B)$ may not be trivial (or approximately trivial). We will use $KL^1(A, M(B)/B)$ to give a classification of full essential extensions up to approximately unitary equivalence.

Proposition 7.2. *Let D be a unital C^*-algebra for which there is a unital embedding from $\mathcal{O}_2$ to D. Let $h_1, h_2 : \mathcal{O}_2 \to D$ be two full homomorphisms. Suppose that $h_1(1_{\mathcal{O}_2}) \sim h_2(1_{\mathcal{O}_2})$. Then there is a sequence of partial isometries v_n such that*

$$v_n^*v_n = h_2(1_{\mathcal{O}_2}),\ \ v_nv_n^* = h_1(1_{\mathcal{O}_2}) \quad \text{and} \quad \lim_{n\to\infty} \|v_n^*h_1(a)v_n - h_2(a)\| = 0 \tag{e 51}$$

for all $a \in \mathcal{O}_2$.

Proof. This is the combination of Theorem 6.5 and Lemma 7.2 in [30]. □

Lemma 7.3. *Let A be a unital separable C^*-algebra, B and C be unital C^*-algebras such that $B \otimes \mathcal{O}_2$ is a unital C^*-subalgebra of C and C has property (P1). Suppose that $h_1, h_2 : A \to B \otimes \mathbb{C} \cdot 1 \subset B \otimes \mathcal{O}_2$ are two unital full monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent in C.*

Proof. It follows from [39] that $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$. Let $p_n = 1_B \otimes q_n \otimes 1_{\mathcal{O}_2}$, where $\{q_n\}$ is a sequence of mutually orthogonal non-zero projections in $\mathcal{O}_2$. Note that $p_n \sim 1_{B\otimes\mathcal{O}_2\otimes\mathcal{O}_2}$, $n = 1, 2, ...$ Define $\phi_i(a) = p_i h_1(a)$ and $\psi_i(a) = p_i h_2(a)$ for all $a \in A$. Also define $\Phi_n(a) = (1 - \sum_{i=1}^n p_i)h_1(a)$ and $\Psi_n(a) = (1 - \sum_{i=1}^n p_i)h_2(a)$ for all $a \in A$. Then, for each n, $h_1 = \sum_{i=1}^n \phi_i \oplus \Phi_n$ and $h_2 = \sum_{i=1}^n \psi_i \oplus \Psi_n$. Note that ϕ_i, Φ_n, ψ_i and Ψ_n are all full. Now we work in $B \otimes \mathcal{O}_2 \otimes 1$. There are partial isometries $v_{i,j} \in \mathcal{O}_2$ such that

$$v_{i,j}^* v_{i,j} = p_j \quad \text{and} \quad v_{i,j}v_{i,j}^* = p_i, \;\; i, j = 1, 2, ..., n \tag{e 52}$$

$$\text{and} \quad v_{n+1,j}^* v_{n+1,j} = p_j, \; v_{n+1,j}v_{n+1,j}^* = 1 - \sum_{i=1}^n p_i, \;\; j = 1, 2, ..., n. \tag{e 53}$$

Put $w_{i,j} = 1 \otimes v_{i,j} \otimes 1$. Then we also have

$$w_{i,1}^* \phi_1 w_{i,1} = \phi_i, \;\; i = 1, 2, ..., n \quad \text{and} \quad w_{n+1,1}^* \phi_1 w_{n+1,1} = \Phi_n. \tag{e 54}$$

Let $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n, ...$ be an increasing sequence of finite subsets of A such that $\cup_{n=1}^\infty \mathcal{F}_n$ is dense in A. It follows from Lemma 5.4.2 of [27] that, for each n, there are isometries $u_n, v_n \in B \otimes \mathcal{O}_2 \otimes 1$ such that

$$\|u_n^* h_1(a)u_n - h_2(a)\| < 1/n \quad \text{and} \quad \|v_n^* h_2(a)v_n - h_1(a)\| < 1/n \;\text{ for }\; a \in \mathcal{F}_n.$$

Note that the relative commutant of $B \otimes \mathcal{O}_2 \otimes 1$ contains a unital C^*-subalgebra $1_B \otimes 1_{\mathcal{O}_2} \otimes \mathcal{O}_2$ which is isomorphic to $\mathcal{O}_2$. It follows from 1.10 in [20] that h_1 and h_2 are approximately unitarily equivalent. □

Lemma 7.4. *Let A be a unital separable nuclear C^*-algebra, B_1, B_2 be two unital C^*-algebra and C be another unital C^*-algebra. Suppose that $j_i : B_i \otimes \mathcal{O}_2 \to C$ are two full monomorphisms so that $j_1(1) \sim j_2(1)$ and $h_i : A \to B_i$ are two full unital monomorphisms. Then there is a sequence of partial isometries $v_n \in C$ such that $v_n^* v_n = j_1(1)$, $v_n v_n^* = j_2(1)$ and*

$$\lim_{n\to\infty} \|v_n^*(j_2 \circ h_2(a))v_n - j_1 \circ h_1(a)\| = 0 \;\text{ for all }\; a \in A. \tag{e 55}$$

Proof. To simplify notation, we may assume that $j_1(1) = j_2(1)$. Therefore we may assume that both j_1 and j_2 are unital. Define $J_i : B \otimes \mathcal{O}_2 \to l^\infty(C)$ by $J_i(b) = (j_i(b), j_i(b), ...)$ for $b \in B_i \otimes \mathcal{O}_2$ and $H_i = J_i \circ h_i$, respectively, $i = 1, 2$. Note that these maps are full in $l^\infty(C)$. Since there is a unital $\mathcal{O}_2$ embedding to $l^\infty(C)$, by 7.2, we obtain unitaries $u_n \in C$ such that

$$\lim_{n\to\infty} \|u_n^* J_2(1 \otimes b)u_n - J_1(1 \otimes b)\| = 0 \;\text{ for all }\; b \in \mathcal{O}_2. \tag{e 56}$$

Denote $U = \{u_n\}$ in $l^\infty(C)$. Let ω be a free ultrafilter on $\mathbb{N}$ and $\pi : l^\infty(C) \to q_\omega(C)$ be the quotient map. Let D be the C^*-subalgebra generated by $\pi \circ J_1(B_1 \otimes \mathbb{C} \cdot 1_{\mathcal{O}_2})$ and $\pi \circ \mathrm{ad}\, U \circ J_2(B_2 \otimes \mathbb{C} \cdot 1_{\mathcal{O}_2})$. It follows that D', the commutant of D, contains $J_1(1_{B_1} \otimes \mathcal{O}_2)$ which is isomorphic to $\mathcal{O}_2$. Therefore we may write $D \subset D \otimes \mathcal{O}_2$. Now $\pi \circ H_1$ and $\pi \circ \mathrm{ad}W \circ H_2$ are two full unital monomorphisms from A into $D \subset D \otimes \mathcal{O}_2$. It follows from 7.3 that $\pi \circ H_1$ and $\pi \circ \mathrm{ad}W \circ H_2$ are approximately unitarily equivalent. It follows from Lemma 6.2.5 of [40] that $j_1 \circ h_1$ and $j_2 \circ h_2$ are approximately unitarily equivalent. □

Theorem 7.5. *Let A be a unital separable nuclear C^*-algebra, B be a unital C^*-algebra which has property (P1), (P2) and (P3). Let $j_o : A \to \mathcal{O}_2 \to B$ be a full embedding of A into B which factors through $\mathcal{O}_2$. Suppose that $\tau : A \to B$ is a full monomorphism. Then there is a sequence of partial isometries $V_n \in M_2(B)$ such that $V_n^* V = 1_B \oplus j_o(1_A)$, $V_n V_n^* = 1_B$ and*

$$\lim_{n\to\infty} \|V_n(\tau \oplus j_o)(a)V_n^* - \tau(a)\| = 0 \;\text{ for all }\; a \in A.$$

Proof. Let $J : B \to l^\infty(B)$ be defined by $J(c) = (c, c, ...)$ for $c \in B$. Define $\tau_\infty = J \circ \tau$ and $J_o = J \circ j_o$. Let ω be a free ultrafilter on $\mathbb{N}$ and $\pi : l^\infty(B) \to q_\omega(B)$ be the quotient map. It follows from 6.5 that $\pi \circ \tau_\infty(A)'$ contains a unital C^*-subalgebra which is isomorphic to $\mathcal{O}_\infty$. Denote this C^*-subalgebra by $\mathcal{O}_\infty$. Let $q \in \mathcal{O}_\infty$ be a nonzero projection such that $[q] = 0$ in $K_0(\mathcal{O}_\infty)$. There is a C^*-subalgebra C of $\mathcal{O}_\infty$ for which $1_C = q$ and $C \cong \mathcal{O}_2$. Put $\tau_0(a) = q\pi \circ \tau_\infty(a)$. So we may view τ_0 is a unital full homomorphism from A into $\tau_0(A) \otimes \mathcal{O}_2$. Since $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ (by [39]), it follows from 7.4 that $\tau_0 \oplus \pi \circ J_o$ and τ_0 are approximately unitarily equivalent. Thus $\pi \circ \tau_\infty$ and $\pi \circ \tau_\infty \oplus \pi \circ J_o$ are approximately unitarily equivalent. It follows from 6.2.5 [40] that τ and $\tau \oplus j_o$ are approximately unitarily equivalent.

□

Proof of Theorem 2.9

Proof. Since A is separable, there is a unital embedding $j : A \to \mathcal{O}_2$, by 2.8 of [20]. Since B has property (P2), there is a full monomorphism $\sigma : \mathcal{O}_2 \to B$. Define $\bar{j} = \sigma \circ j$. Note $\bar{j}$ is full. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. It follows from Theorem 3.9 of [28] that there is an integer n and a unitary $v \in M_{n+1}(B)$ such that

$$\|v^*\mathrm{diag}(h_1(a), \bar{j}(a), \bar{j}(a), ..., \bar{j}(a))v - \mathrm{diag}(h_2(a), \bar{j}(a), \bar{j}(a), ..., \bar{j}(a))\| < \varepsilon/4 \qquad \text{(e 57)}$$

for all $a \in \mathcal{F}$. On the other hand, by 7.2, there is an isometry $u \in M_n(\pi \circ \sigma(\mathcal{O}_2))$ with $uu^* = 1_{M(C)/C}$ such that

$$\|u^*\bar{j}(a)u - \mathrm{diag}(\bar{j}(a), \bar{j}(a), ..., \bar{j}(a))\| < \varepsilon/4 \qquad \text{(e 58)}$$

for $a \in \mathcal{F}$. Thus, we obtain an isometry $w \in M_2(B)$ with $ww^* = 1_B$ such that

$$\|w^*\mathrm{diag}(h_1(a), j(a))w - \mathrm{diag}(h_2, j(a))\| < \varepsilon/2 \ \text{ for all } \ a \in \mathcal{F}. \qquad \text{(e 59)}$$

By applying 7.5, we obtain a partial isometry $z \in B$ such that $z^*h_1(1_A)z = h_2(1_A)$, $zh_2(1_A)z^* = h_1(1_A)$ and

$$\|z^*h_1(a)z - h_2(a)\| < \varepsilon \ \text{ for all } \ a \in \mathcal{F}. \qquad \text{(e 60)}$$

□

Remark 7.6. If both h_1 and h_2 are unital, it is clear that z can be chosen to be unitary. If one of them is unital and the other is not, z can never be unitary. Suppose that both are not unital. Since B has property (P1),(P2) and (P3), we obtain full $\mathcal{O}_2$ embeddings into $h_1(1_A)Bh_1(1_A)$ and $h_2(1_A)Bh_2(1_A)$. Therefore there is a projection $e \le h_1(1_A)$ such that $h_1(1_A)$ is equivalent to $h_1(1_A) - e$ and e is a full projection. So there is a partial isometry $v \in B$ such that $v^*v = h_1(1_A)$ and $vv^* = h_1(1_A) - e$. Thus $1 - \mathrm{ad}\, v^* \circ h_1(1_A)$ is full. Similarly, there is a partial isometry $w \in B$ with $w^*w = h_2(1_A)$ such that $1 - \mathrm{ad}\, w^* \circ h_2(1_A)$ is full. Now apply 2.9 to the case that $A = \mathbb{C}$. we know that $1 - \mathrm{ad}v^* \circ h_1(1_A)$ and $1 - \mathrm{ad}w^* \circ h_2(1_A)$ are equivalent. This implies that we can choose z to be unitary in the proof of 2.9.

Corollary 7.7. *Theorem 2.9 also holds for the case that $B = q_\infty(\{C_n\})$, where each C_n is a unital purely infinite simple C^*-algebras.*

Proof. It is clear that B has property (P1) and (P2). From the proof of 2.9 above, we only need an absorbing lemma 7.5 for this B. Let $\tau : A \to B$ be a full monomorphism and $j_0 : A \to \mathcal{O}_2 \to B$ be a full embedding of A into B which factors through $\mathcal{O}_2$. So we may write $j_0 = \Phi \circ j$, where $j : A \to \mathcal{O}_2$ is a monomorphism and $\Phi : \mathcal{O}_2 \to B$ is a full homomorphism. Let $L : A \to l^\infty(\{C_n\}$ be a contractive completely positive linear map for which $\pi \circ L = \tau$, where $\pi : l^\infty(\{C_n\}) \to q_\infty(\{C_n\})$ is the quotient map. Write $L = \{L_n\}$, where $L_n : A \to C_n$ is a contractive completely positive linear map. Let $\phi_n : \mathcal{O}_2 \to C$ such that $\pi \circ \{\psi_n\} = \Phi$. Denote by D_n the separable unital purely infinite simple C^*-algebra containing $L_n(A)$ and $\psi_n(\mathcal{O}_2)$. Then $q_\infty(\{D_n\}) \subset B$ and $\tau : A \to q_\infty(\{C_n\})$ and $j_0 : A \to \mathcal{O}_2 \to q_\infty(\{C_n\})$. Thus one applies 7.5 of [29]. $\square$

Proof of Proposition 2.12

Proof. Let $h_1 : A \to B \otimes \mathcal{K}$ be a homomorphism. It follows from 4.5 in [29] that there is a sequence of asymptotically multiplicative contractive completely positive linear maps $\{\phi_n\}$ from A to $B \otimes \mathcal{K}$ and a sequence of unitaries $u_n \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\lim_{n\to\infty} \|(h \oplus \phi_n)(a) - \mathrm{ad}\, u_n \circ j(a)\| = 0 \ \text{ for all } \ a \in A. \tag{e 61}$$

Since B has property (P2), it is easy to see that we may assume that ϕ_n maps A into B and u_n are unitaries in B. It follows from 6.5 in [30] that, for each k, there exists a sequence of unitaries $v_n(k) \in M_2(B)$ such that

$$\lim_{n\to\infty} \|v_n(k)^*(\phi_n(a) \oplus j_o(a))v_n(k) - (\phi_{n+k}(a) \oplus j_o(a))\| = 0 \ \text{ for all } \ a \in A. \tag{e 62}$$

It follows from 4.7 of [29] that there exists a homomorphism $h_1 : A \to M_2(B)$ and a sequence of unitaries $w_n \in M_2(B)$ such that

$$\lim_{n\to\infty} \|\mathrm{ad}\, w_n \circ h_1(a) - (\phi_n(a) \oplus j_o(a))\| = 0 \ \text{ for all } \ a \in A. \tag{e 63}$$

By applying the fact that B has property (P2) and applying 7.2, we obtain a sequence of isometries $z_n \in M_3(B)$ with $z_n z_n^* = j_o(1_A)$ such that

$$\lim_{n\to\infty} \|(h \oplus h_1 \oplus j_o)(a) - z_n^* j_o(a) z_n\| = 0 \ \text{ for all } \ a \in A. \tag{e 64}$$

It follows that $[h_1] = -[h]$ in $H(A, B)$. $\square$

Proof of 2.13

Proof. The corollary follows immediately from 2.12 and 7.5. $\square$

8 Classification of full extensions

Definition 8.1. Let C_n be a commutative C^*-algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Put $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$ (see [41]). One has the following six-term exact sequence (see [41]):

$$\begin{array}{ccccc} K_0(A) & \to & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \to & K_1(A) \\ \uparrow_{\mathbf{k}} & & & & \downarrow_{\mathbf{k}} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A)\,. \end{array}$$

In [12], $K_i(A, \mathbb{Z}/n\mathbb{Z})$ is identified with $KK^i(\mathbb{I}_n, A)$ for $i = 0, 1$. As in [12], we use the notation

$$\underline{K}(A) = \oplus_{i=0,1,n\in\mathbb{Z}_+} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-called Bockstein operations (see [12]). It follows from the definition in [12] that if $x \in KK(A, B)$, then the Kasparov product $KK^i(\mathbb{I}_n, A) \times x$ gives an element in $KK^i(\mathbb{I}_n, B)$ which we identify with $\mathrm{Hom}(K_i(A, \mathbb{Z}/n\mathbb{Z}), K_0(B, \mathbb{Z}/n\mathbb{Z}))$. Thus one obtains a map $\Gamma : KK(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$. It is shown by Dadarlat and Loring ([12]) that if A is in $\mathcal{N}$ then, for any σ-unital C^*-algebra B, the map Γ is surjective and $\ker \Gamma = Pext(K_*(A), K_*(B))$. In particular,

$$\Gamma : KL(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

is an isomorphism. It is shown in [28] that if A satisfies AUCT, then Γ is also an isomorphism from $KL(A, B)$ onto $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.

Lemma 8.2. *Let B be a unital C^*-algebra which admits a full $\mathcal{O}_2$ embedding and let G_i be a countable subgroup of $K_i(B)$ $(i = 0, 1)$. There exists a unital separable C^*-algebra $B_0 \subset B$ which has a full $\mathcal{O}_2$ embedding such that $K_i(B_0) \supset G_i$ and $j_{*i} = \mathrm{id}_{K_i(B_0)}$, where $j : B_0 \to B$ is the embedding.*

Proof. Let $p_1, ..., p_n, ...$ be projections and $u_1, u_2, ..., u_n, ...$ be unitaries in $\cup_{k=1}^\infty M_k(B)$ such that $\{p_n\}$ and $\{u_n\}$ generates of G_0 and G_1, respectively. There is a countable set S such that

$$p_n, u_n \in \cup_{n=1}^\infty \{(a_{ij})_{n\times n} \in M_n(B) : a_{ij} \in S\}$$

Let $j_o : \mathcal{O}_2 \to B$ be a full embedding. Let $p = j(1_{\mathcal{O}_2})$ and $x_1, x_2, ..., x_m \in B$ such that $\sum_{i=1}^m x_i^* p x_i = 1$. Let B_1 be the unital separable C^*-subalgebra generated by S, $\{x_1, x_2, ..., x_m\}$ and $j(\mathcal{O}_2)$. Then B_1 has a full $\mathcal{O}_2$ embedding and $p_n, u_n \in \cup_{k=1}^\infty M_k(B_1)$ for all n. Note that $K_i(B_1)$ is countable. The embedding $j_1 : B_1 \to B$ gives homomorphisms $(j_1)_{*i} : K_i(B_1) \to K_i(B)$. Let $F_{1,i}$ be the subgroup of $K_0(B_1)$ generated by $\{p_n\}$ and $\{u_n\}$, respectively. It is clear that $(j_1)_{*i}$ is injective on $F_{1,i}$, $i = 0, 1$. In particular, the image of $(j_1)_{*i}$ contains G_i, $i = 0, 1$. Let $N'_{1,i} = \ker(j_1)_{*i}$ and let $N_{1,i}$ be the set of all projections (if $i = 0$), or unitaries (if $i = 1$) in $\cup_{k=1}^\infty M_k(B_1)$ which have images in $N'_{1,i}$. Let $\{p_{1,n}\}$ be a dense subset of projections in $\cup_{k=1}^\infty M_k(B_1)$. There are countable pairs of projections $\{e_n, e'_n\}$ in $\{p_{1,n}\}$ such that $[e_n] = [e'_n]$ in $K_0(B)$. There are $w_n \in \cup_{k=1}^\infty M_k(B)$ such that $w_n^* w_n = e_n \oplus 1_{k(n)}$ and $w_n w_n^* = e'_n \oplus 1_{k(n)}$.

Let $\{u_{1,n}\}$ be a dense subset of unitaries in $\cup_{k=1}^\infty M_k(B_1)$. For each $u_{1,n}$, there are unitaries $z_{1,n,k} \in \cup_{j=1}^\infty M_j(B)$, $k = 1, 2, ..., m(n)$ such that

$$\|z_{1,n,1} - 1\| < 1/2, \|z_{1,n,m(n)} - u_{1,n}\| < 1/2 \quad \text{and} \quad \|z_{1,n,k} - z_{1,n,k+1}\| < 1/2,$$

$k = 1, 2, ..., m(n)$, $n = 1, 2, ...$ Let B_2 be a separable unital C^*-algebra containing B_1 such that $\cup_{k=1}^\infty M_k(B_2)$ contains all $\{w_{1,n}\}$ and $\{z_{1,n,k}\}$. Note that there is a full embedding of $\mathcal{O}_2$ to B_2. Note also that if $p, q \in \cup_{k=1}^\infty M_k(B_1)$ are projections so that $[p] - [q] \in N_{1,0}$ then $[p] - [q] = 0$ in $K_0(B_2)$. Similarly, if $u \in B_1$ and $[u] \in N_{1,1}$, then $[u] = 0$ in B_2. Suppose that B_l has been constructed. Let $j_l : B_l \to B$ be the embedding. Let $N_{l,i} = \ker(j_l)_{*i}$, $i = 0, 1$. As before, we obtain a unital separable C^*-algebra $B_{l+1} \supset B_l$ such that every pair projections p, $q \in \cup_{k=1}^\infty M_k(B_l)$ with $[p] - [q] \in N_{l,0}$ has the property that $[p] = [q]$ in $K_0(B_{l+1})$, and every unitary $u \in B_l$ with $[u] \in N_{l,1}$ has the property that $[u] = 0$ in $K_1(B_{l+1})$. Let B_0 be the closure of $\bigcup_{l=1}^\infty B_l$. Note B_0 admits a full $\mathcal{O}_2$ embedding. Note also that B_0 is separable. Let $j : B_0 \to B$ be the embedding.

We claim that j_{*i} is injective. Suppose that $p\,, q \in M_k(B_0)$ is a pair of projections for which $[p]-[q] \in \ker j_{*0}$ and $[p]-[q] \neq 0$ in B_0. Without loss of generality, we may assume that $p\,, q \in M_k(B_l)$ for some large integer l. Then $[p]-[q]$ must be in the $\ker(j_l)_{*0}$. By the construction, $[p]-[q]=0$ in $K_0(B_{l+1})$. This would imply that $[p]-[q]=0$ in $K_0(B_0)$. Thus j_{*0} is injective. An exactly same argument shows that j_{*1} is also injective. The lemma then follows. $\square$

Lemma 8.3. *Let B be a unital C^*-algebra which admits a full $\mathcal{O}_2$ embedding. Suppose that $G_i \subset K_i(B)$ and $F_i(k) \subset K_i(B, \mathbb{Z}/k\mathbb{Z})$ are countable subgroups such that the image of $F_i(k)$ in $K_{i-1}(B)$ is contained in G_{i-1} $(i=0,1, k=2,3,...)$. Then there exists a separable unital C^*-algebra $C \subset B$ which admits a full $\mathcal{O}_2$ embedding such that $K_i(C) \supset G_i$, $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset F_i(k)$ and the embedding $j : C \to B$ induces an injective map $j_{*i} : K_i(C) \to K_i(B)$ and an injective map $j_* : K_i(C, \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z})$, $k=2,3,....$*

Proof. It follows from 8.2 that there is a separable unital C^*-algebra C_1 which admits a full $\mathcal{O}_2$ embedding such that $K_0(C_1) \supset G_0$ and $K_1(C_1) \supset G_1$ and j induces an identity map on $K_0(C_1)$ and $K_1(C_1)$, where $j : C_1 \to B$ is the embedding. Fix k, and let $\{x \in K_i(C_1) : kx = 0\} = \{g_1^{(i)}, g_2^{(i)}, ...,\}$. Suppose that $\{s_1^{(i)}, s_2^{(i)}, ...,\}$ is a subset of $K_{i-1}(B, \mathbb{Z}/k\mathbb{Z})$ such that the map from $K_{i-1}(B, \mathbb{Z}/k\mathbb{Z})$ to $K_i(B)$ maps $s_j^{(i)}$ to $g_j^{(i)}$. For each $z^{(i)} \in K_{i-1}(C_1, \mathbb{Z}/k\mathbb{Z})$, there is $s_j^{(i)}$ such that $z^{(i)} - s_j^{(i)} \in K_i(B)/kK_i(B)$. Since $K_i(C_1)$ is countable, the set of all possible $z^{(i)} - s_j^{(i)}$ is countable. Thus one obtains a countable subgroup G_i' which contains $K_i(C_1)$ for which $G_i'/kK_i(B)$ contains the above the mentioned countable set as well as $F_i(k) \cap (K_i(B)/kK_i(B))$ for each k. Since countably many countable sets is still countable, we obtain a countable subgroup $G_i^{(2)} \subset K_i(B)$ such that $G_i^{(2)}$ contains G_i' and $kK_i(B) \cap G_i^{(2)} = kG_i^{(2)}$, $k=1,2,...,$ and $i=0,1$. Note also $F_i(k) \cap (K_i(B)/kK_i(B)) \subset G_i^{(2)}/kK_i(B)$. By applying 8.2, we obtain a separable unital C^*-algebra $C_2 \supset C_1$ such that $K_i(C_2) \supset G_i^{(2)}$ and an embedding from C_2 to B gives an injective map on $K_i(C_2)$, $i=0,1$. Repeating what we have done above, we obtain an increasing sequence of countable subgroups $G_i^{(n)} \subset K_i(B)$ such that $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)}$ for all k and $i=0,1$ and an increasing sequence of separable C^*-subalgebras C_n such that $K_i(C_n) \supset G_i^{(n)}$ and embeddings from C_n into B giving injective maps on $K_i(C_n)$, $i=0,1$, and $n=1,2,....$ Moreover $F_i^{(k)} \cap (K_i(B)/kK_i(B)) \subset K_i(C_n)/kK_i(B)$. Let C denote the closure of $\bigcup_n C_n$ and $j : C \to B$ be the embedding. Then C is a separable unital C^*-algebra and j_{*i} is an injective map, $i=0,1$. Since $C \supset C_1$ and C_1 is unital, C admits a full $\mathcal{O}_2$ embedding. We claim that $K_i(C) \cap kK_i(B) = kK_i(C)$, $k=1,2,...,$ and $i=0,1$. Note that $K_i(C) = \cup_n G_i^{(n)}$. Since $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)} \subset kK_i(C)$, we see that $K_i(C) \cap kK_i(B) = kK_i(C)$, $i=0,1$. Thus $K_i(C)/kK_i(C) = K_i(C)/kK_i(B)$. Since $K_i(C)/kK_i(B) \supset F_i^{(k)} \cap (K_i(B)/kK_0(B))$, we conclude also that $K_i(C, \mathbb{Z}/k\mathbb{Z})$ contains $F_i(k)$. Since j_{*i} is injective, j induces an injective map from $K_i(C)/kK_i(C)$ into $K_i(B)/kK_i(B)$ for all integer $k \geq 1$. Using this fact and the fact that $j_{*i} : K_i(C) \to K_i(B)$ is injective, by chasing the following commutative diagram

$$\begin{tikzcd}
K_0(C) \arrow[rr] \arrow[dr, "j_{*0}"] & & K_0(C,\mathbf{Z}/k\mathbf{Z}) \arrow[rr] \arrow[d, "j_*"'] & & K_1(C) \arrow[dl, "j_{*1}"'] \arrow[ddd] \\
& K_0(B) \arrow[r] & K_0(B,\mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(B) \arrow[d] & \\
& K_0(B) \arrow[u] & K_1(B,\mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(B) \arrow[l] & \\
K_0(C) \arrow[uuu] \arrow[ur, "j_{*0}"] & & K_1(C,\mathbf{Z}/k\mathbf{Z}) \arrow[ll] \arrow[u, "j_*"] & & K_1(C) \arrow[ll] \arrow[ul, "j_{*1}"']
\end{tikzcd}$$

one sees that j induces an injective map from $K_i(C,\mathbb{Z}/k\mathbb{Z})$ to $K_i(B,\mathbb{Z}/k\mathbb{Z})$. $\square$

Corollary 8.4. *Without assuming that B has a full $\mathcal{O}_2$ embedding, both 8.3 and 8.2 hold if we do not require that C (or B_0) has a full $\mathcal{O}_2$ embedding.*

Proof of Theorem 2.16

Proof. By 2.8, it suffices to show that, for each $x \in KL(A, M(B)/B)$, there is a full monomorphism $h : A \to M(B)/B$ such that $[h] = x$. Put $Q = M(B)/B$. Since A satisfies the AUCT, we may view x as an element in $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q))$. Note that $K_i(A)$ is a countable abelian group ($i = 0, 1$). Let $G_0^{(i)} = \gamma(x)(K_i(A))$, $i = 0, 1$, where $\gamma : \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q)) \to \mathrm{Hom}(K_*(A), K_*(Q))$ is the surjective map. Then $G_0^{(i)}$ is a countable subgroup of $K_i(Q)$, $i = 0, 1$. Consider the following commutative diagram:

$$\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr, "\gamma(x)"] & & K_0(A,\mathbf{Z}/k\mathbf{Z}) \arrow[rr] \arrow[d, "\times x"'] & & K_1(A) \arrow[dl, "\gamma(x)"'] \arrow[ddd] \\
& K_0(Q) \arrow[r] & K_0(Q,\mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(Q) \arrow[d] & \\
& K_0(Q) \arrow[u] & K_1(Q,\mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(Q) \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur, "\gamma(x)"] & & K_1(A,\mathbf{Z}/k\mathbf{Z}) \arrow[ll] \arrow[u, "\times x"] & & K_1(A) \arrow[ll] \arrow[ul, "\gamma(x)"']
\end{tikzcd}$$

It follows from 8.3 that there is a unital C^*-algebra $C \subset Q$ which has a full $\mathcal{O}_2$ embedding such that $K_i(C) \subset G_0^{(i)}$, $K_i(C) \cap kK_i(Q) = kK_i(C)$, $k = 1, 2, ...$, and $i = 0, 1$, and the embedding $j : C \to Q$ induces injective maps on $K_i(C)$ as well as on $K_i(C, \mathbb{Z}/k\mathbb{Z})$ for all k and $i = 0, 1$. Moreover $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset (\times x)(K_i(A, \mathbb{Z}/k\mathbb{Z}))$ for $k = 1, 2, ...$ and $i = 0, 1$. We have the following commutative diagram:

$$\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr, "\gamma(x)"] & & K_0(A,\mathbf{Z}/k\mathbf{Z}) \arrow[rr] & & K_1(A) \arrow[dl, "\gamma(x)"'] \arrow[ddd] \\
& K_0(C) \arrow[r] & K_0(C,\mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(C) \arrow[d] & \\
& K_0(C) \arrow[u] & K_1(C,\mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(C) \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur, "\gamma(x)"] & & K_1(A,\mathbf{Z}/k\mathbf{Z}) \arrow[ll] & & K_1(A) \arrow[ll] \arrow[ul, "\gamma(x)"']
\end{tikzcd}$$

We will add two more maps on the above diagram. From the fact that the image of $K_i(A, \mathbb{Z}/k\mathbb{Z})$ under $\times x$ is contained in $K_i(C, \mathbb{Z}/k\mathbb{Z})$, ($k = 2, 3, ...$, $i = 0, 1$), we obtain two maps $\beta_i : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_i(C, \mathbb{Z}/k\mathbb{Z})$,

$k = 2, 3, ..., i = 0, 1$ such that $j_* \circ \beta_i = \times x$ and obtain the following commutative diagram:

$$\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr, "\gamma(x)"] & & K_0(A, \mathbf{Z}/k\mathbf{Z}) \arrow[rr] \arrow[d, "\beta_0"'] & & K_1(A) \arrow[dl, "\gamma(x)"'] \arrow[ddd] \\
& K_0(Q) \arrow[r] & K_0(C, \mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(C) \arrow[d] & \\
& K_0(C) \arrow[u] & K_1(C, \mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(C) \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur, "\gamma(x)"] & & K_1(A, \mathbf{Z}/k\mathbf{Z}) \arrow[ll] \arrow[u, "\beta_1"] & & K_1(A) \arrow[ll] \arrow[ul, "\gamma(x)"']
\end{tikzcd}$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\to & K_i(A, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(A, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(A, \mathbb{Z}/m\mathbb{Z}) & \to \\
& \downarrow & & \downarrow & & \downarrow & \\
\to & K_i(Q, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(Q, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(Q, \mathbb{Z}/m\mathbb{Z}) & \to
\end{array}$$

Since $j_* \circ \beta_i = \times x$ and all vertical maps in the following diagram is injective

$$\begin{array}{ccccccccc}
\to & K_i(C, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(C, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(C, \mathbb{Z}/m\mathbb{Z}) & \to \\
& \downarrow & & \downarrow & & \downarrow & \\
\to & K_i(Q, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(Q, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(Q, \mathbb{Z}/m\mathbb{Z}) & \to
\end{array}$$

,

we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
\to & K_i(A, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(A, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(A, \mathbb{Z}/m\mathbb{Z}) & \to \\
& \downarrow & & \downarrow & & \downarrow & \\
\to & K_i(C, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(C, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(C, \mathbb{Z}/m\mathbb{Z}) & \to
\end{array}$$

Thus we obtain an element $y \in KL(A, C)$ such that $y \times [j] = x$. Since A satisfies the AUCT, one checks that $KL(A, C) = KL(A \otimes \mathcal{O}_\infty, C)$. This also follows from the fact that the unital embedding from $A \to A \otimes \mathcal{O}_\infty$ gives a KK-equivalence (see [34]). It follows from 6.6 and 6.7 in [30] that there exists a homomorphism $\phi : A \otimes \mathcal{O}_\infty \to C \otimes \mathcal{K}$ such that $[\phi] = y$. Define $\psi = \phi|_{A \otimes 1}$. By the same result of Pimsner ([34]), one obtains that $[\psi] = y$. Since A is unital, we may assume that the image of ψ is in $M_m(C)$ for some integer $m \geq 1$. Since C admits a full $\mathcal{O}_2$ embedding, C has property (P2). Thus 1_m is equivalent to a projection in C. Thus we may further assume that ψ maps A into C. Put $h_1 = j \circ \psi$. To obtain a full monomorphism, we note that there is an embedding $\imath : A \to \mathcal{O}_2$ (see Theorem 2.8 in [20]). Since $M(B)/B$ has property (P2), we obtain a full monomorphism $\psi : \mathcal{O}_2 \to M(B)/B$. Let $e = \psi(1_{\mathcal{O}_2})$. There is a partial isometry $w \in M_2(M(B)/B)$ such that $w^* w = 1_{M(B)/B}$ and $ww^* = 1 \oplus e$. Define $h = w^*(h_1 \oplus \psi \circ \imath)w$. One checks that $[h] = [h_1] = x$ and h is a full monomorphism. $\square$

Corollary 8.5. *Let A be a unital separable amenable C^*-algebra satisfying the AUCT. Let B be a unital C^*-algebra which has property (P2). Then, for each $x \in KL(A, B)$, there is a full monomorphism $h : A \to B$ such that $[h] = x$.*

Proof. In the proof above, we may replace $M(B)/B$ by B. $\square$

Proof of Theorem 2.17

Proof. For the first part of the theorem, it suffices to show that every essential full extension is absorbing. Let τ be a such extension. Following Elliott and Kocerovsky, we will show that τ is purely large. Denote $E = \tau^{-1}(A)$. Choose $c \in E \setminus C$. Then, by 3.3, c is a full element. Since $M(C)$ has property (P1), there exists $x \in M(B)$ such that $x^*cc^*x = 1$. Therefore there exists a projection $p \leq cc^*$ for which there is $v \in M(B)$ such that $v^*v = 1$ and $vv^* = p$. Note $\overline{cBc^*} = \overline{cM(B)c^*} \cap B$. So $pBp \subset \overline{cBc^*}$. Now $v^*pBpv = B$. So pBp is stable and pBp is full. Thus τ is purely large. So it is absorbing. The last part of the theorem follows from the next corollary. $\square$

Corollary 8.6. *Let A be a separable unital amenable C^*-algebra, C be a unital C^*-algebra and $B = C \otimes \mathcal{K}$. Then $Ext(A, B)$ is the same set as unitary equivalence classes of essential full extensions of A by B.*

Proof. It suffices to show that given any element $x \in Ext(A, B)$, there exists an essential full extension $\tau : A \to M(B)/B$ so that $[\tau] = x$. There exists a $\tau_1 : A \to M(B)/B$ such that $[\tau_1] = x$. Take a monomorphism $j : A \to \mathcal{O}_2$ (see [20]). Let $h : \mathcal{O}_2 \to M(\mathcal{K})$ be a monomorphism (given by a faithful representation of $\mathcal{O}_2$ on a separable Hilbert space). Let $\phi : M(\mathcal{K}) \to M(B)$ be the standard unital embedding and $\pi : M(B) \to M(B)/B$ be the quotient map. Then $\tau_2 = \pi \circ \phi \circ h \circ j$ gives a full essential trivial extension. It follows that $\tau = \tau_1 \oplus \tau_2$ is an essential full extension. Since $[\tau_2] = 0$, $[\tau] = [\tau_1] = x$. $\square$

Remark 8.7. Let B be a non-stable, non-unital but σ-unital C^*-algebra. Suppose that $M(B)/B$ has property (P1), (P2) and (P3), and suppose that $\tau : A \to M(B)/B$ is an essential full extension. One should not expect that such extension is purely large in general. Let $0 \to B \to E \to A \to 0$ be an essential full extension corresponding to τ. Recall that the extension is purely large if cBc^* contains a C^*-subalgebra which is stable and cBc^* is full in B (see [15]). Given any element $c \in E \setminus B$, $\pi(c)$ is full in $M(B)/B$. But, in general, c need not be full in $M(B)$, nor doescBc^* need to be full in B. Examples are easily seen in the case that $B = c_0(C)$, where C is a unital purely infinite simple C^*-algebra. Suppose that $0 \to c_0(C) \to E \to A \to 0$ is a full extension and $c' \in E \setminus c_0(C)$. Write $c' = \{c'_n\} \in l^\infty(C)$. Define $c_n = c'_n$ if $n \geq N > 1$ and $c_n = 0$ if $n \leq N$. Put $c = \{c_n\}$. Then $c \in E \setminus c_0(C)$. However, it is clear that $cc_0(C)c^*$ is not full in $c_0(C)$. By 7.5, the full extension τ is approximately absorbing in the sense of 7.5 but not purely large. It should be also noted that, even if c^*Bc is full for all $c \in E \setminus B$, the full extension may not be purely large. Let B be a nonstable, non-unital but σ-unital simple C^*-algebra with continuous scale (see[31] for more examples). Then B may be stably finite. No hereditary C^*-subalgebra of B contains a stable C^*-subalgebra. So none of the essential extensions of a unital separable amenable C^*-algebra A by B could be possibly purely large in the sense of [15], nevertheless, all of these extensions are approximately absorbing in the sense of 7.5 (and many of them are actually absorbing; for example, when $A = C(X)$).

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